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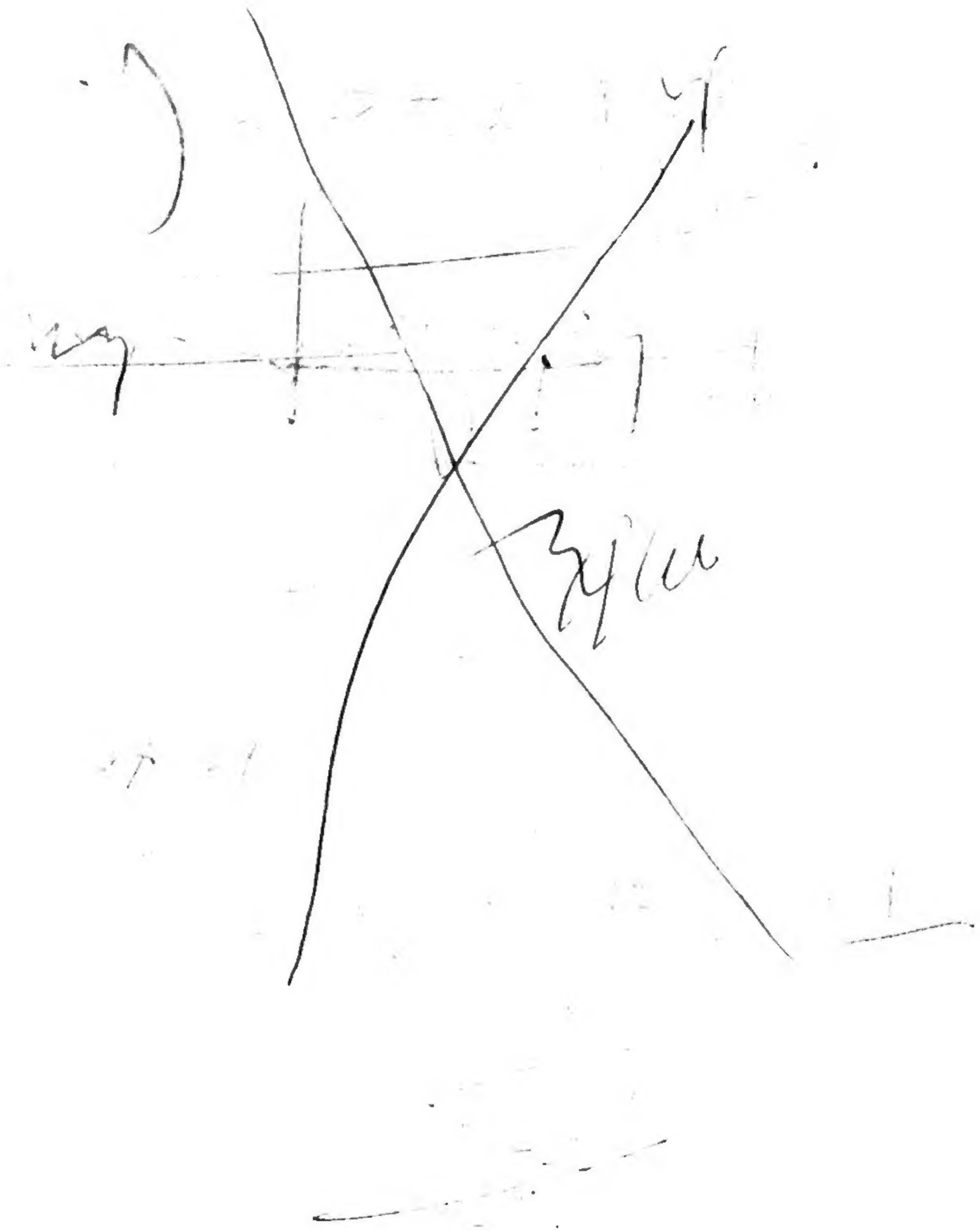

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Gulabsons'

Intermediate Algebra

BY

M. A. RAHMAN, M. A.
Lecturer in Mathematics
Mohindra College
PATIALA

PT. S. N. KAUL, M. A.
Lecturer in Mathematics
S. P. College
SRINAGAR (Kashmir)

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PREFACE

By producing this book the authors do not mean to swarm the number of Algebra books for the Intermediate Students, but their aim is to place the subject before the desiring student in a becoming manner and natural order.

In this book the student will find a separate chapter on the Simultaneous Quadratic Equations. These equations, although a requirement of the day, are not dealt with, so for, in other books on the subject. Here a sufficiently thorough study of these equations is given.

Partial Fractions are treated in their various aspects earlier than usually is done. A knowledge of these fractions is required to sum certain type of Series. So they are discussed before the Series. Special care is paid to the fact that the general theory of the Partial Fractions is not required by the Intermediate Student.

At the end of each topic a set of miscellaneous questions is given wherein are included more difficult problems for the curious student to advance his knowledge. To these questions, in some cases, hints for solution are appended.

Sfficient pains have been taken to free the book from mistakes and mis-prints. But the authors are human, as are the publishers. If, therefore, the teachers point out any mistakes and mis-prints or suggest any improvements, the authors will feel indebted.

In the end the authors thank Mr. M. Abdullah who helped much in proof-reading.

The Authors.

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Gulabsons'

Intermediate Algebra

CHAPTER I.

Quadratic Equations.

1. Definition. An equation which *when simplified to a rational integral form* contains terms involving an unknown quantity upto the second degree only is called a *quadratic equation*, or equation of the second degree.

For example

$$2x^2+5x-2=0\ ;\ 9x^2-4=0\ ;\ 3x^2+4=0\ ;$$

$$2x+\frac{3}{x}+1=0\ ;\ \ x-\sqrt{x}+2=0\ ;\ x^2+2abx=b^2+2ab^2$$

are all quadratic equations.

Here x is the unknown quantity ; the digits and the letters a, b are the known quantities. Generally the unknown quantity is denoted by one of the letters x, y, z ; and the known quantities are the digits and the letters a, b, c,...

In an equation the unknown quantity is called a *variable* and the known quantity, a *constant*. The digits 1, 2, 3.....are called *arithmetical constants* ; and the letters a, b, c,.....*algebraical constants*.

2. Solution of Quadratic Equation. The values of the variable for which an equation is satisfied are called the *roots* of that equation. These roots can always be expressed in terms of the constants occuring in the

equation. The process of finding the roots is called *solving* the equation.

The equation $ax^2+bx+c=0$ we shall always take as the standard from of a quadratic equation. Clearly this will be a quadratic equation when $a\neq 0$.

There are two methods of solving a quadratic equation.

3. First Method *Factor Method.* We know that

$$(lx+m)(x+n)=lx^2+(m+ln)x+mn.$$

If we can express ax^2+bx+c in the form of $lx^2+(m+ln)x+mn$, then the equation $ax^2+bx+c=0$ is at once solved. For, then

$$ax^2+bx+c=lx^2+(m+ln)x+mn$$
$$=(lx+m)(x+n)=0.$$

Now the product of two factors being zero, either of them may be zero. If $lx+m=0$, then $x=-\frac{m}{l}$; and if $x+n=0$, then $x=-n$.

Thus $-\frac{m}{l}$, $-n$ are the two values of x.

Ex. 1. Solve $2x^2+x-6=0$.

Here $2x^2+x-6=2x^2+(-3+4)x-6$
$$=(2x-3)(x+2).$$

Thus the given equation is the same as

$$(2x-3)(x+2)=0.$$

Now if $2x-3=0$, then $x=\frac{3}{2}$;
and if $x+2=0$, then $x=-2$.

Hence the two values of x are $\frac{3}{2}$, -2.

Note. If we substitute either of these roots in the given equation, we see that it is satisfied. Thus we can verify our solution.

Ex. 2. Solve $x^2+5x-14=0$.

The given equation is the same as

$$x^2+(7-2)x-14=0$$

or $$(x+7)(x-2)=0$$

$\therefore$ $$x+7=0 \text{ or } x-2=0$$

Hence $x=-7$ or 2.

Ex. 3. Solve $\frac{a}{ax-1}+\frac{b}{bx-1}=a+b$. [P. U. 1922]

Transposing we get $\left(\frac{a}{ax-1}-b\right)+\left(\frac{b}{bx-1}-a\right)=0$

or $\frac{a+b-abx}{ax-1}+\frac{a+b-abx}{bx-1}=0$

or $(a+b-abx)\left[\frac{1}{ax-1}+\frac{1}{bx-1}\right]=0.$

$\therefore$ either $a+b-abx=0$, whence $x=\frac{a+b}{ab}$

$=\frac{1}{a}+\frac{1}{b}$,

or $\frac{1}{ax-1}+\frac{1}{bx-1}=0$, whence $\frac{bx-1+ax-1}{(ax-1)(bx-1)}=0$

or $(a+b)x-2=0$

or $x=\frac{2}{a+b}$.

Hence the two roots are $\frac{1}{a}+\frac{1}{b}$, $\frac{2}{a+b}$.

4. Second Method. *Completing the Square.* Since all expressions of the second degree are not always easy to factorize, we shall solve the standard equation in the general form.

The equation $ax^2+bx+c=0$,
on division by a, which is not zero, and on transposition becomes

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

or $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$

or $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$.

Taking the square root of both sides, we have

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Note. As the constants a, b, c, may have any numerical values, we can say that we have solved the quadratic equation in the most general form. The solution of any particular equation can be obtained by substituting the numerical values of a, b, c from the given equation in the formula (1).

Ex. 1. Solve $2x^2-x-6=0$.

(*i*) By completing the square :

Dividing by 2, the given equation is

$$x^2-\tfrac{1}{2}x-3=0$$

or $x^2-\frac{1}{2}x+(\frac{1}{4})^2=3+(\frac{1}{4})^2$, on completing the square

or $(x-\frac{1}{4})^2=\frac{49}{16}$

or $(x-\frac{1}{4})^2=(\frac{7}{4})^2$

Taking the square root, we have $x-\frac{1}{4}=\pm\frac{7}{4}$

Hence, $x=2$ and $-\frac{3}{2}$.

(*ii*) By making use of the formula (1)

Here we see that $a=2$, $b=-1$, $c=-6$.

Substituting these values in $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, we get

$$x=\frac{1\pm\sqrt{1+48}}{4}=\frac{1\pm7}{4}.$$

$$\therefore \quad x=2 \text{ or } -\tfrac{3}{2}.$$

Ex. 2. Solve $\dfrac{x+2}{x-3}+\dfrac{x+1}{x-2}=4$.

The equation is the same as

$$\left(1+\frac{5}{x-3}\right)+\left(1+\frac{3}{x-2}\right)=4$$

or $$\frac{5}{x-3}+\frac{3}{x-2}=2$$

or $$\frac{5(x-2)+3(x-3)}{(x-3)(x-2)}=2$$

or $8x-19=2(x-3)(x-2)=2x^2-10x+12$

or $2x^2-18x+31=0$.

Here $a=2$, $b=-18$, $c=31$.

$$\therefore \quad x=\frac{18\pm\sqrt{18^2-248}}{4}=\frac{18\pm\sqrt{4.19}}{4}$$

$$= \frac{18 \pm 2\sqrt{19}}{4} = \frac{9 \pm \sqrt{19}}{2}.$$

Thus $x = \frac{9+\sqrt{19}}{2}$ or $\frac{9-\sqrt{19}}{2}$.

Ex. 3. Solve $a(x^2+1) = x(a^2+4)$. (P. U. 1926)

Transposing, $ax^2 - (a^2+4)x + a = 0$.

Here $a = a$, $b = -(a^2+4)$, $c = a$

$$\therefore \quad x = \frac{(a^2+4) \pm \sqrt{(a^2+4)^2 - 4a^2}}{2a}$$

$$= \frac{(a^2+4) \pm \sqrt{a^4+4a^2+16}}{2a}$$

Hence the two roots of x are

$$\frac{a^2+4+\sqrt{a^4+4a^2+16}}{2a},$$

and $$\frac{a^2+4-\sqrt{a^4+4a^2+16}}{2a}.$$

Exercise I.

Solve by factors the following equations :—

1. $3x^2 - 16x + 5 = 0$.

2. $x^2 - 12x + 35 = 0$.

3. $x^2 + 3x - 28 = 0$.

4 $6x^2 + 5x - 6 = 0$.

5. $49x^2 - 21x - 10 = 0$.

6. $x^2 - 4x - 165 = 0$.

7. $8x^2 - 17x - 21 = 0$.

8. $9x^2 - 19x + 2 = 0$.

9. $\frac{a}{x-b} + \frac{b}{x-a} = 2$.

10. $x^2 - 2px - 1 = -p^2$.

11. $x^2 - 10ax + 21a^2 = 0$.

12. $acx^2 - \left[a(a+b) + c(b+c) \right]x + (a+b)(b+c) = 0$.

Solve by formula the following equation :—

13. $x^2 - 20x + 51 = 0$.

14. $x^2 - 2x - 4 = 0$.

15. $x^2 - 2ax + a^2 - 2 = 0$.

16. $c(x^2+1) = x(c^2+1)$.

17. $\frac{1}{8+x} + \frac{1}{5-x} = \frac{13}{42}$.

18. $x + \frac{1}{x} = 3\frac{1}{3}$.

(P. U. 1934)

19. $\frac{x+3}{x} + \frac{x}{x+3} = \frac{13}{6}$.

20. $\left(\frac{x+p}{x-p}\right)^2 - 2\left(\frac{x+p}{x-p}\right) - 35 = 0$.

21. $\frac{x+2}{x-1}+2\frac{2}{3}=\frac{x+3}{x-2}$. (P. U. 1934)

22. $\frac{x+2}{x-3}-\frac{1}{x-1}=3$.

23. $\frac{1}{x-m+l}-\frac{1}{x-m-l}=\frac{2l}{l^2-n^2}$.

24. $\frac{x-4}{x-6}-\frac{x-2}{x-21}=\frac{2}{x+13}$.

25. $lx+\frac{m^2-n^2}{lx}=2m$.

26. $\sqrt{x^2-2x-2}=3$.

27. $\sqrt{5x^2-14x+10}=x-3$.

28. $\sqrt{(c^2+1)x^2-2(b+ac)x+a^2}=b-x$.

29. $\frac{\sqrt{2+x^2}+\sqrt{2-x^2}}{\sqrt{2+x^2}-\sqrt{2-x^2}}=\frac{\sqrt{2}+1}{\sqrt{2}-1}$.

30. $\frac{1}{\sqrt{1-x}+1}+\frac{1}{\sqrt{1+x}-1}=\frac{1}{x}$.

31. $\frac{x-b}{x-a}-\frac{b}{a}=\frac{a}{b}-\frac{x-a}{x-b}$.

32. $(a+x)^{\frac{2}{3}}-(a-x)^{\frac{2}{3}}=(a^2-x^2)^{\frac{1}{3}}$.

33. $x=\cfrac{2}{3-\cfrac{2}{3-\cfrac{2}{3-x}}}$

34. $x=\cfrac{5}{6-\cfrac{5}{6-\cfrac{5}{6-x}}}$

35. The height h at any time t, of a body thrown upwards with the velocity u, is given by the formula $h=ut-\frac{1}{2}gt^2$. If $u=+0$ ft. per second and the value of g is 32, find the time when the body is at a height of 24 ft. Account for the double time.

36. A straight line 6 inches in length is divided into two parts such that the square on one part equals the product of the whole line and the other part. Find the length of the first part. What is the reason for the double answer ?

37. A number decreased by unity is equal to 20 times its reciprocal, find it.

5. Imaginary Quantities. The qauntities dealt with so far are such as can be measured out. For instance, there is no exact arithmetical expression giving the values of the numbers like $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, etc. Yet the same can be geometrically represented by straight lines. Similarly a quantity like $-\sqrt{4}$ can be indicated by a straight line. Such like quantities are *real quantities* and are the result of solving the quadratic equation $x^2=a$, a being a positive number, which gives for x two roots $\pm\sqrt{a}$.

Now the corresponding equation $x^2=-a$, a being positive, will give two roots $\pm\sqrt{-a}$. These roots suggest the quantity of the type $\sqrt{-5}$. Arithmetically the quantity $\sqrt{-5}$ is meaningless, for it is the quantity which when squared gives us -5, and there is no arithmetical process to obtain such a quantity. Also, geometrically we cannot represent $\sqrt{-5}$ by a straight line as we can represent the corresponding number $\sqrt{5}$. Again, negative numbers have algebraical significance only; their square roots cannot be interpreted arithmetically. Such quantities are, therefore, called *imaginary quantities.*

We shall assume that imaginary numbers obey the fundamental laws of algebra.

As $\sqrt{-a}=\sqrt{(-1)a}=\sqrt{a}\times\sqrt{-1}$, it is evident that every imaginary quantity is the product of two factors of which one is always $\sqrt{-1}$ and the other is real. The factor $\sqrt{-1}$ is often denoted by i. Thus the presence of i with any quantity as a factor means that the quantity is imaginary. So $5i$, $i\sqrt{3}$, $\sqrt{ia}$, $-i\sqrt{b}$ are all imaginary quantities.

The powers of i can be easily established. For

$$\begin{aligned} i &= \sqrt{-1} \\ i^2=(\sqrt{-1})^2 &= -1 \\ i^3=i^2.i &= -i \\ i^4=i^2.\,i^2 &= 1. \end{aligned}$$

By the help of these four powers of i, any other power of i can easily be determined. For instance, $i^{54}=i^{52}.i^2=(i^4)^{13}.i^2=i^2=-1$, and $i^{4n}=(i^4)^n=1^n=1$.

6. Definitions. The algebraic sum of a real and an imaginary quantity is called a *complex quantity*. For example $5+i\sqrt{3}$, $2-i$, $a+ib$, $p-i\sqrt{q}$ are all complex quantities.

When in any two complex quantities, the imaginary parts differ in sign only the complex quantities are said to be *conjugate*. Thus $a+ib$ and $a-ib$ are conjugate quantities.

The numbers of the type 2 and $\frac{1}{2}$; $\frac{3}{4}$ and $\frac{4}{3}$; a and $\frac{1}{a}$; $\frac{1}{m}$ and m are called the *reciprocal numbers*. The reciprocal of *zero* is *infinity* (∞). As any multiple of zero, is zero so any multiple of ∞ is infinity. Thus $\frac{16}{0}=16\times\frac{1}{0}$. $\therefore \frac{16}{0}$ is ∞; and $\frac{m}{0}=m\times\frac{1}{0}$, $\therefore \frac{m}{0}$ is ∞. For further discussion on zero and infinity, see appendix.

7. We shall now give some properties of complex quantities.

I. If $a+ib=0$, then $a=0$ and $b=0$.

For, $a+ib=0$

$\therefore \quad a=-ib$

$\therefore \quad a^2=-b^2$

$\therefore \quad a^2+b^2=0.$

Now the sum of two positive quantities cannot be zero unless each is zero. Hence $a=0$ and $b=0$.

II. If $a+ib=c+id$, then $a=c$ and $b=d$.

For, $a+ib=c+id$

$\therefore \quad a-c=i(d-b)$

$\therefore \quad (a-c)^2=-(d-b)^2$

$\therefore \quad (a-c)^2+(d-b)^2=0.$

Whence, we have as before $a-c=0$, $d-b=0$.

Hence $a=c$, and $d=b$.

Cor. If $a+ib=c+id$, then $a-ib=c-id$.

III. The sum and difference of two complex quantities are complex.

For, $(a+ib)+(c+id)=(a+c)+i(b+d)$

and $(a+ib)-(c+id)=(a-c)+i(b-d)$.

IV. The product and quotient of two complex quantities are complex.

For, $(a+ib)(c+id)=ac+iad+ibc+i^2bd$

$=(ac-bd)+i(ad+bc)$;

and $\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}$.

Cor. The product of two conjugate complex quantities is always real.

8. *To find the cube-roots* of *unity.*

Suppose that x is a value of $(1)^{\frac{1}{3}}$. Then $x=(1)^{\frac{1}{3}}$, *i.e.*

$x^3=1$.

or $x^3-1=0$, or $(x-1)(x^2+x+1)=0$,

whence, either $x=1$, or $x^2+x+1=0$

$\therefore x=1$ or $x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$.

The cube roots of unity, therefore, are

$1, \frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$.

9. *Properties of the imaginary cube-roots of unity.*

I. One imaginary cube root is the square of the other imaginary cube root.

For, $\left\{\frac{-1+i\sqrt{3}}{2}\right\}^2=\frac{1-3-2i\sqrt{3}}{4}=\frac{-1-i\sqrt{3}}{2}$.

Thus if one is denoted by w, the other will be w^2.

II. The sum of the two imaginary cube roots of unity is -1.

For, $w^2+w=\frac{-1+i\sqrt{3}}{2}+\frac{-1-i\sqrt{3}}{2}=\frac{-2}{2}=-1$.

Thus we get the very important relation $w^2+w+1=0$.

III. The two imaginary cube roots are reciprocals of each other.

For, $\dfrac{1}{w}=\dfrac{2}{-1+i\sqrt{3}}=\dfrac{2(-1-i\sqrt{3})}{(-1+i\sqrt{3})(-1-i\sqrt{3})}$

$=\dfrac{2(-1-i\sqrt{3})}{1+3}=\dfrac{-1-i\sqrt{3}}{2}=w^2.$

Cor. $w^3=1.$

Solved Examples

Ex. 1. Show that

$(1+2w+w^2)(1+w+2w^2)=1.$

L. H. S. $=1+w+2w^2+2w+2w^2+4w^3+w^2+w^3+2w^4$

$=1+3w+5w^2+5+2w,$ (since $w^4=w\cdot w^3=w$)

$=1+5(1+w+w^2)$

$=1,$ $1+w+w^2$ being equal to zero.

Ex. 2. Prove that

$(x^3+y^3)=(x+y)(x+wy)(x+w^2y)$

R. H. S. $=(x^2+wxy+xy+wy^2)(x+w^2y)$

$=x^3+wx^2y+x^2y+wxy^2+w^2yx^2+w^3xy^2$

$+w^2xy^2+w^3y^3$

$=x^3+x^2y(1+w+w^2)+xy^2(1+w+w^2)+y^3$

$=x^3+y^3.$

Ex. 3. Find the square root of $-3+4i.$

Let $\sqrt{-3+4i}=x+yi$

Squaring both sides, we get

$-3+4i=x^2-y^2+2xyi,$

whence, $x^2-y^2=-3$.....(i)

$xy=2$..... (ii)

By trial we see that $x=1$, $y=2$ satisfies the equations (i) and (ii)

$\therefore$ The required square root is $1+2i.$

EXERCISE II

Show that :—

1. $(2+i)(3-i)=7+i$ **2.** $(1+2i)(1-3i)=7-i.$

3. $(2+3i)(-1-i)=1-i5.$

4. $\dfrac{4+5i}{4-5i}=-\frac{9}{41}+i\frac{40}{41}.$

5 $\dfrac{l-im}{l+im}=\dfrac{l^2-m^2}{l^2+m^2}-i\dfrac{2lm}{l^2+m^2}.$

Express the following in the form $A+iB$:—

6. $(9-2i)(2+9i)$. **7.** $(a+ib)(b+ia)$.

8. $\dfrac{2+3i}{3+2i}$. **9.** $\dfrac{1}{1-i}$.

10. $\dfrac{3+2i}{3+4i}+\dfrac{4+3i}{3-4i}$. **11.** $\dfrac{p+iq}{p-iq}-\dfrac{p-iq}{p+iq}$.

12. Show that $\dfrac{2+3i+4i^2}{2-3i+4i^2}=-\dfrac{5}{13}-i\dfrac{12}{13}$.

Prove that

13. $a^2+1=(a+i)(a-i)$.

14. $x^2+9=(x+i3)(x-i3)$.

15. $x^2+y^2=(x+iy)(x-iy)=(ix+y)(-ix+y)$.

16. $25=(4+i3)(4-i3)$.

17. Show that $\pm i$ and ± 1 are all the roots of the equation $x^4-1=0$.

If w and w^2 are the imaginary cube roots of unity, show that

18. $(1+w)(1+w^2)=1$.

19. $(1-w)(1-w^2)=3$.

20. $(1+w-w^2)(1-w+w^2)=4$.

21. $(1-w+w^2)^3=-8$.

22. $(1-w)(1-w^2)(1-w^3)=0$.

23. $w^{16}+w^8+w^{-2}\cdot w^{-4}=0$.

24. Show that the imaginary roots of the equation $x^3+1=0$ are $-w$ and $-w^2$.

Prove that :—

25. $x^2+xy+y^2=(x-wy)(x-w^2y)$.

26. $a^3+b^3=(a+b)(a+wb)(a+w^2b)$.

Find the square root of :—

27. $-5+12i$. **28.** $9-i40$.

29. i. **30.** $-4i$.

Factorise :—

31. x^2+y^2. **32.** x^2+4.

33. Show that $x^3+1=(x+1)(x+w)(x+w^2)$.

10. Equations reducible to the Quadratic Form :—

There are many types of equations which can be solved by reducing them to the quadratic equations.

Ex. 1. Solve $3x^4+4x^2-4=0$.

Instead of x if we consider x^2 as the variable, then the equation is a quadratic in x^2. Thus

$$x^2=\frac{-4\pm\sqrt{16+48}}{6}=\frac{-4\pm8}{6}=-2 \text{ or } \frac{2}{3}.$$

$x^2=-2$ gives $x=\pm i\sqrt{2}$, and $x^2=\frac{2}{3}$ gives $x=\pm\sqrt{\frac{2}{3}}$.

Or Put $x^2=y$. Then the given equation becomes $3y^2+4y-4=0$, etc.

Ex. 2. Solve $x^6-19x^3-216=0$.

Here consider the equation as a quadratic in x^3. Then

$$x^3=\frac{19\pm\sqrt{361+864}}{2}=\frac{19\pm35}{2}=27 \text{ or } -8.$$

When $x^3=27$, we get $x=3,\ 3w,\ 3w^2$;
and when $x^3=-8$, we get $x=-2,\ -2w,\ -2w^2$.

Or Put $x^3=y$ and proceed.

Ex. 3. Solve for x the equation.

$$\frac{x-2}{x+3}-28\left(\frac{x+3}{x-2}\right)-3=0.$$

Instead of x, here consider the fraction $\frac{x-2}{x+3}$ as the variable. Then from the given equation

$$\frac{x-2}{x+3}-\frac{28}{x-2}-3=0$$

$$x+3$$

or $$\left(\frac{x-2}{x+3}\right)^2-3\left(\frac{x-2}{x+3}\right)-28=0.$$

It is a quadratic equation in $\frac{x-2}{x+3}$.

Thus $$\frac{x-2}{x+3}=\frac{3\pm\sqrt{9+112}}{2}=7 \text{ or } -4.$$

When $\frac{x-2}{x+3}=7$, we get $x=\frac{23}{6}$, and when $\frac{x-2}{x+3}=-4$ we get $x=-2$.

Or Put $\frac{x-2}{x+3}=y$, then the equation becomes

$$y-28\frac{1}{y}-3=0, \text{ etc.}$$

Ex. 4. Solve $2^{2x}-3.2^{x+2}+32=0$.

The given equation is $2^{2x}-3.2^{x}.2^{2}+32=0$

or $$(2^x)^2-12.2^x+32=0.$$

It is a quadratic equation in 2^x. Thus

$$2^x=\frac{12\pm\sqrt{144-128}}{2}=\frac{12\pm4}{2}=8 \text{ or } 4.$$

When $2^x=8=2^3$, we get $x=3$; and when $2^x=4=2^2$, we get $x=2$.

Ex. 5. Solve $(x+3)(x+4)(x+5)(x+6)=15$.

The equation is $(x+3)(x+6)(x+4)(x+5)=15$.

or $$(x^2+9x+18)(x^2+9x+20)=15$$

Put $x^2+9x+18=y$, then $y(y+2)=15$

or $y^2+2y-15=0$ whence $y=3$ *or* -5.

When $y=3$, we have $x^2+9x+18=3$ or $x^2+9x+15=0$

giving $x=\frac{-9\pm\sqrt{21}}{2}$.

When $y=-5$, we have $x^2+9x+18=-5$

or $x^2+9x+23=0$ giving $x=\frac{-9\pm\sqrt{-11}}{2}$

Thus the roots are $\frac{-9\pm\sqrt{21}}{2}, \frac{-9\pm\sqrt{-11}}{2}$.

Ex. 6. Solve $3x^2-\sqrt{3x^2+2x-1}=7-2x$.

Re-arranging the terms, $3x^2+2x-1-\sqrt{3x^2+2x-1}=6$.

Put $\sqrt{3x^2+2x-1}=y$, then the equation becomes $y^2-y-6=0$, giving $y=3$ or -2.

When $y=3$, we have $\sqrt{3x^2+2x-1}=3$

or $$3x^2+2x-1=9.$$

or $$3x^2+2x-10=0;$$

whence $x=\frac{-2\pm\sqrt{4+120}}{6}=\frac{-1\pm\sqrt{31}}{3}$;

and when $y=-2$, we have $\sqrt{3x^2+2x-1}=-2$

or $3x^2+2x-1=4$ or $3x^2+2x-5=0$;

whence $x=\frac{-2\pm\sqrt{4+60}}{6}=\frac{-2\pm8}{6}=\frac{-5}{3}$ or $+1$.

By actual substitution we see that $x=\frac{-1\pm\sqrt{31}}{3}$ satisfy the given equation, while $x=1$, and $-\frac{5}{3}$ do not.

Hence the roots are $\frac{-1\pm\sqrt{31}}{3}$.

Note. It may be remarked that $1, -\frac{5}{3}$ are the roots of $3x^2+\sqrt{3x^2+2x-1}=7-2x$.

Ex. 7. Solve $\sqrt{3x^2-2x+9}+\sqrt{3x^2-2x-4}=13$.

Since for all values of x, we have

$$(3x^2-2x+9)-(3x^2-2x-4)=13 \quad \ldots\ldots\ldots(1)$$

We get on dividing (1) by the given equation

$$\sqrt{3x^2-2x+9}-\sqrt{3x^2-2x-4}=1 \quad \ldots\ldots\ldots(2)$$

Now adding the given equation to (2), we get

$$\sqrt{3x^2-2x+9}=7$$

or $3x^2-2x+9=49$

or $3x^2-2x-40=0$, whence $x=-\frac{10}{3}$ or 4.

Ex. 8. Solve $4x^4-16x^3+23x^2-16x+4=0$.

Re-arranging the terms, $4(x^4+1)-16x(x^2+1)+23x^2=0$.

Dividing by x^2, $\quad 4\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+23=0$

or $\quad 4\left\{\left(x+\frac{1}{x}\right)^2-2\right\}-16\left(x+\frac{1}{x}\right)+23=0$

or $\quad 4\left(x+\frac{1}{x}\right)^2-16\left(x+\frac{1}{x}\right)+15=0 \quad \ldots\ldots\ldots(1)$

In this equation consider $x+\frac{1}{x}$ as the variable.

Then $x+\frac{1}{x}=\frac{16\pm\sqrt{16^2-16\times15}}{8}=\frac{5}{2}$ or $\frac{3}{2}$.

When $x+\frac{1}{x}=\frac{5}{2}$, we get $2x^2-5x+2=0$,

$$\therefore \quad x=2 \text{ or } \frac{1}{2};$$

and when $x+\frac{1}{x}=\frac{3}{2}$, we get $2x^2-3x+2=0$

$$\therefore \quad x=\frac{3\pm i\sqrt{7}}{4}.$$

Hence the roots are $2, \frac{1}{2}, \frac{3\pm i\sqrt{7}}{4}$.

Or In (1) put $x+\frac{1}{x}=y$, then it becomes

$$+y^2-16y+15=0, \text{ etc.}$$

Note 1. Notice that in the last example the coefficients of terms equidistant from the beginning and end are equal in magnitude. Such like equations are called *reciprocal equations*. They are so called because a reciprocal equation remains unaltered if $\frac{1}{x}$ is substituted for x. The equidistant terms may be (*i*) equal or (*ii*) opposite in sign. The equations belonging to the latter class may be solved in a similar manner.

In a reciprocal equation of the even degree belonging to class (*ii*), the middle term must be wanting. $x=\pm 1$ will be seen to be its two roots. If, however, the middle term be not absent, the roots will occur in pairs, each pair being reciprocals of each other but of opposite sign.

Note 2. In reciprocal equations of the odd degree, if the equidistant coefficients are equal, $x=-1$ is a root and if the equidistant coefficients are equal in magnitude but opposite in sign, $x=1$ is a root.

Ex. 9. $8x^4-42x^3+29x^2+42x+8=0.$

The equation belongs to class (*ii*) and the middle term is not wanting.

Dividing by x^2, we get

$$8x^2-42x+29+42\cdot\frac{1}{x}+\frac{8}{x^2}=0$$

or $$8\left(x^2+\frac{1}{x^2}\right)-42\left(x-\frac{1}{x}\right)+29=0.$$

Now putting $x-\frac{1}{x}=y$, we have

$$x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2x\cdot\frac{1}{x}=y^2+2.$$

$\therefore$ The given equation transforms into

$$8(y^2+2)-42y+29=0$$

or $$8y^2-42y+45=0$$

Hence $y=3/2$ and $y=15/4$

i.e., $x-\frac{1}{x}=3/2$; and $x-\frac{1}{x}=15/4$

whence $x=2$ or $-\frac{1}{2}$; and $x=4$ or $-\frac{1}{4}$.

Ex. 10. Solve—

$$\sqrt{2x^2+5x-7}+\sqrt{3(x^2-7x+6)}=\sqrt{7x^2-6x-1}.$$

We have from the equation

$$\sqrt{(2x+7)(x-1)}+\sqrt{3(x-6)(x-1)}=\sqrt{(7x+1)(x-1)}.$$

Here we get $\sqrt{x-1}=0$, *i.e.*, $x=1$

or $\sqrt{2x+7}+\sqrt{3(x-6)}=\sqrt{7x+1}$.

Squaring and transposing, we get

$$2\sqrt{3(2x+7)(x-6)}=2(x+6)$$

or $$3(2x+7)(x-6)=(x+6)^2$$

or $$5x^2-27x-162=0, \text{ giving } x=9 \text{ or } -\tfrac{18}{5}.$$

On substitution we find that $-\frac{18}{5}$ does not satisfy the equation.

Hence the roots are 1 and 9.

EXERCISE III.

Solve for x the following equations :—

1. $x^4-28x^2+75=0$.

2. $12x^4-x^2-1=0$.

3. $8x^6-65x^3+8=0$.

4. $x+5x^{\frac{1}{2}}+6=0$.

5. $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-3=0$.

6. $x^{\frac{1}{4}}+x^{\frac{1}{2}}-6=0$.

7. $x^{\frac{2}{n}}+ax^{\frac{1}{n}}-12a^2=0$.

8. $3^{2x}-10.3^x+9=0$.

9. $3^{2x}-4.3^{x+2}+243=0$.

10. $5^x+5^{1-x}=26$.

11. $7^{-x}+7^{1+x}-8=0$.

12. $\frac{1-x}{x}+\frac{x}{1-x}=2\frac{1}{2}$.

13. $\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}=2\frac{1}{6}$.

14. $\sqrt{\dfrac{2x^2+1}{3x^2-1}}+6\sqrt{\dfrac{3x^2-1}{2x^2+1}}=5.$

15. $\sqrt{\dfrac{2x^2+3x+1}{5x^2-2x+3}}+2\sqrt{\dfrac{5x^2-2x+3}{2x^2+3x+1}}=3.$

16. $\sqrt{3^x}+\dfrac{9}{\sqrt{3^x}}=10.$ **17.** $(x+1)(x+2)(x+3)(x+4)=15.$

18. $(x+4)(x-5)(x+6)(x-7)=504.$

19. $x(2x+1)(x-2)(2x-3)=63.$

20. $(2x+5)(2x-7)(x^2-9)=91.$

21. $x^2-3x-5\sqrt{x^2-3x+6}=0.$

22. $x^2-2x+3=2\sqrt{x^2-2x+2}.$

23. $9\sqrt{3x^2-7x+2}=3x^2-7x-8.$

24. $x^2-\dfrac{x}{2}-7=x-3\sqrt{2x^2-3x+2}.$

25. $2(x^2-3x+1)^2+5(x^2-3x+1)+3=0.$

26. $x^2-21=\sqrt{x^2-9}.$

27. $\sqrt{3x^2+7x-1}-\sqrt{3x^2+7x-10}=1.$

28. $\sqrt{5x^2-6x+8}+\sqrt{5x^2-6x-7}=15.$

29. $\sqrt{3x^2-7x-30}-\sqrt{2x^2-7x-5}=x-5.$

30. $\sqrt{x^2+x+4}+\sqrt{x^2-x+3}=2x+1.$

31. $x^4+x^3-4x^2+x+1=0.$ **32.** $x^4-3x^3-2x^2-3x+1=0.$

33. $4x^4-4x^3+5x^2-4x+4=0.$

34. $x^4-5x^3+8x^2-5x+1=0.$

35. $6x^4-25x^3+12x^2+25x+6=0.$

36. $x^2+\dfrac{1}{x^2}+2\left(x+\dfrac{1}{x}\right)=\dfrac{142}{9}.$

37. $\sqrt{2x^2-9x+4}+3\sqrt{2x-1}=\sqrt{2x^2+21x-11}.$

38. $\sqrt{x^2-8x+15}-\sqrt{4x^2-18x+18}+\sqrt{x^2+2x-15}=0.$

39. $\sqrt{x+2}+\sqrt{2x+1}=\sqrt{3x+5}.$

40. $\sqrt{3x+1}-\sqrt{3x+2}=3\sqrt{x+1}.$

41. $\sqrt{2x+7}+\sqrt{7x+2}=3\sqrt{x+1}.$

42. $\sqrt{x+4}+\sqrt{x+7}=\sqrt{x+11}$.

43. $\dfrac{2-\sqrt{4x-x^2}}{2+\sqrt{4x-x^2}}=\dfrac{x}{2-x}$ [apply componendo and dividendo]

44. $\dfrac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}}=\dfrac{5x}{a}$.

45. $3^{x^2}:3^{2x}=27:1$. **46.** $n^{2x}(n^2+1)=(n^{3x}+n^x)n$.

47. $ax+a(x^2-ax+b^2)^{\frac{1}{2}}-x^2=ab$.

48. $\sqrt{1+\dfrac{x}{a}}-\sqrt{1-\dfrac{a}{x}}=1$.

49. $\dfrac{x^2}{\sqrt{a}+\sqrt{b}}-(\sqrt{a}-\sqrt{b})x=\dfrac{1}{\dfrac{1}{b\sqrt{a}}+\dfrac{1}{a\sqrt{b}}}$.

50. $\binom{x}{x-1}^2+\binom{x}{x+1}^2=n(n-1)$.

The left-hand side $\binom{x}{x-1}+\binom{x}{x+1}^2-\dfrac{2x^2}{x^2-1}$

$=\binom{2x^2}{x^2-1}^2-\dfrac{2x}{x^2-1}$

Now put $\dfrac{2x^2}{x^2-1}=y$ and proceed.

CHAPTER II.

Theory of Quadratic Equations.

11. Definitions. (*i*) *Equation* : An equation is a statement of eauality between two expressions which holds only for the particul ir values of the variable or variables involved. For example, $3x-9=5x-21$ is an equation, because it is true for only one value of $x=6$.

The sign $=$ is used to denote equality in an equation.

(*ii*) *Identity* An identity is a statement of equality between two expressions which holds for all values of the variable or variables involved For example, $x^2-2ax+a^2=(x-a)^2$ is true for all values of x and a, and, therefore, it is an identity.

The sign $\equiv$ is used to denote the equality in an identity to distinguish it from $=$ used in an equation. Thus $(a+b)(a-b)\equiv a^2-b^2$.

Properties of Identities :

(1) In an identity, the coefficients of the like powers of the variable involved are equal on both the sides.

(2) If a polynomial in x of the form $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$ vanishes for $x=a$, then $(x-a)$ is a factor of the polynomial.

Dividing the polynomial by $x-a$, let Q be the quotient and R be the remainder, where R contains no x. Then

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=(x-a)Q+R. \quad (1)$$

Putting in (1) $x=a$ we have

$$a_0a^n+a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n=0\times Q+R$$
$$=R.$$

But $a_0a^n+a_1a^{n-1}+a_2a^{n-2}+\ldots+a_n$ vanishes for $x=a$,

$\therefore$ R$=0$.

Hence, $x-a$ divides the polynomial exactly, so that $x=a$ is a factor.

12. A Quadratic Equation cannot have more than two roots. If possible, let the general quadratic

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equation $ax^2+bx+c=0$ have three *different* roots, α, β, λ. Then we must have

$$a\alpha^2+b\alpha+c=0 \quad \ldots\ldots\ldots\ldots(1)$$
$$a\beta^2+b\beta+c=0 \quad \ldots\ldots\ldots\ldots(2)$$
$$a\lambda^2+b\lambda+c=0 \quad \ldots\ldots\ldots\ldots(3)$$

Subtracting (2) from (1) we get $a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$

or $$(\alpha-\beta)\{a(\alpha+\beta)+b\}=0.$$

Here, since α and β are different, the factor $\alpha-\beta$ is not zero ;

$$\therefore \quad a(\alpha+\beta)+b=0 \quad \ldots\ldots\ldots\ldots(4)$$

Similary from (2) and (3) we get

$$a(\beta+\lambda)+b=0 \quad \ldots\ldots\ldots\ldots(5)$$

Now subtract (5) from (4). Then

$$a(\alpha-\lambda)=0$$

But a is not zero (2). So $\alpha-\lambda$ must be zero. Thus α and λ cannot be *different* as supposed. Hence there can be *only* two roots.

Cor. 1. If the equation $ax^2+bx+c=0$ is satisfied by three *different* roots, then each of the quantities a, b, c, is zero. For, when α, β, λ are all different, then the relation $a(\alpha-\lambda)=0$ is satisfied only when $a=0$. And $a=0$ gives from (4) or (5), $b=0$. Finally from any of (1), (2), (3) we get $c=0$.

Cor. 2. If the equation $ax^2+bx+c=0$ is satisfied by three *different* values of x, then it is satisfied by *any value* of x. (*i. e.* in that case the equation is an identity). For, the equation now becomes $0.x^2+0.x+0=0$ which is evidently satisfied by all values of x.

13. Relations between the roots and the coefficients. Suppose α, β are the two roots of $ax^2+bx+c=0$. Then evidently

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

$$\text{Now } \alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

$$=\frac{-2b}{2a}=-\frac{b}{a};$$

and $\alpha\beta = \frac{(-b)^2-(b^2-4ac)}{4a^2} = \frac{c}{a}$.

Hence we have the relations :—

(1) Sum of the roots $=$ $\frac{\text{the coefficient of } x \text{ with sign changed}}{\text{the coefficient of } x^2}$;

and (2) Product of the roots $=$ $\frac{\text{constant term}}{\text{the coefficient of } x^2}$

Cor. If α, β be the roots of the equation $ax^2+bx+c=0$, show that for all values of x, $ax^2+bx+c=a(x-\alpha)(x-\beta)$ [P. U. 1927]

Since α, β are the roots, we have

$$\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

$$\text{Now } ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$= a\{x^2-(\alpha+\beta)x+\alpha\beta\}$$
$$= a(x-\alpha)(x-\beta)$$

14. To form the equation with given roots. From above we see that the equation $ax^2+bx+c=0$

is the same as $x^2+\frac{b}{a}x+\frac{c}{a}=0$, since $a\neq 0$.

i. e., $x^2-(\alpha+\beta)x+\alpha\beta=0$.....(1)

or $x^2-(\text{sum of the roots})x+\text{product of the roots}=0$.

i. e. $(x-\alpha)(x-\beta)=0$.....(2)

From (1) we infer that to form the quadratic equation whose roots are given we have to make unity as the coefficient of x^2, put the sum of the roots with sign changed, as the coefficient of x, and the product of the roots as the constant term ; and then the sum of all these three is to be equated to zero.

From (2) we conclude that if we subtract each given root from x and equate to zero the product of the two differences, we get the quadratic equation.

Ex. 1. Find the equation whose roots are 3 and $-\frac{3}{5}$.
From (2) the required equation is $(x-3)(x+\frac{3}{5})=0$.
or $x^2-\frac{12}{5}x-\frac{9}{5}=0$.
or $5x^2-12x-9=0$.

Ex. 2 What equation has the roots $7\pm2\sqrt{5}$?
Here the sum of the roots $=14$
and the product of the roots $=7^2-(2\sqrt{5})^2$
$=49-20=29$.
Hence from (1) the required equation is
$x^2-14x+29=0$.

15. Symmetric functions of the roots. Expressions involving symmetrically the roots of a quadratic equation can be evaluated in terms of the coefficients.

Ex. 1. If α, β be the roots of $ax^2+bx+c=0$, find the value of (1) $\alpha^2+\beta^2$, (2) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.

Since $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$ we have

(1) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2\frac{c}{a}=\frac{b^2-2ac}{a^2}$

and (2) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}=\frac{b^2-2ac}{a^2}\div\frac{c^2}{a^2}=\frac{b^2-2ac}{c^2}$.

Ex. 2. If α, β be the roots of $x^2+lx+m=0$, form the equation whose roots are $\frac{1}{a\alpha+b}$ and $\frac{1}{a\beta+b}$.

As $\alpha+\beta=-l$ and $\alpha\beta=m$, we have for the new equation the sum of the roots $=\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}$

$=\frac{a(\alpha+\beta)+2b}{a^2\alpha\beta+ab(\alpha+\beta)+b^2}=\frac{2b-al}{a^2m-abl+b^2}$;

and the product of the roots $=\frac{1}{(a\alpha+b)(a\beta+b)}$

$=\frac{1}{a^2m-abl+b^2}$.

Hence the required equation is

$$x^2-\frac{2b-al}{a^2m-abl+b^2}x+\frac{1}{a^2m-abl+b^2}=0$$

or $(a^2m-abl+b^2)x^2-(2b-al)x+1=0.$

Ex. 3. Find the condition that one root of $ax^2+bx+c=0$ may be the square of the other. P. U. 1916.

If α and α^2 be the roots then we have

the sum of the roots $=\alpha+\alpha^2=-\frac{b}{a}$ .....(1)

and the product $=\alpha^3=\frac{c}{a}$ .....(2)

From (2) $\alpha=\left(\frac{c}{a}\right)^{\frac{1}{3}}$

$\therefore$ (1) gives $\left(\frac{c}{a}\right)^{\frac{1}{3}}+\left(\frac{c}{a}\right)^{\frac{2}{3}}=-\frac{b}{a}$,

which on cubing and simplifying gives

$$a^2c+ac^2+b^3=3abc.$$

16. Common root of two equations. *To find the condition that the equations* $ax^2+bx+c=0$ *and* $a'x^2+b'x+c'=0$ *may have a common root.*

Suppose α is a root of both the equations. Then we have

$$\left.\begin{aligned}a\alpha^2+b\alpha+c&=0\\ \text{and}\quad a'\alpha^2+b'\alpha+c'&=0\end{aligned}\right\}$$

By the method of cross multiplication,

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b}\quad\text{.....(1)}$$

whence we get $(ca'-c'a)^2=(bc'-b'c)(ab'-a'b)$ as the required condition.

Cor. 1. From (1) it is clear that α, the common root is $\frac{ca'-c'a}{ab'-a'b}$ or $\frac{bc'-b'c}{ca'-c'a}$.

Cor. 2. To find the uncommon roots we note that the product of the roots of the first equation is $\frac{c}{a}$. Therefore the other root of the first equation is $\frac{c}{a}\div\frac{ca'-c'a}{ab'-a'b}$

or $\dfrac{c}{a} \div \dfrac{bc'-b'c}{ca'-c'a}$; *i.e.* $\dfrac{c(ab'-a'b)}{a(ca'-c'a)}$ or $\dfrac{c(ca'-c'a)}{a(bc'-b'c)}$.

Similarly the other root of the second equation is

$$\frac{c'(ab'-a'b)}{a'(ca'-c'a)} \text{ or } \frac{c'(ca'-c'a)}{a'(bc'-b'c)}.$$

Ex. 1. If the equations $x^2+bx+ca=0$ and $x^2+cx+ab=0$ have a common root, their other roots satisfy the equation $x^2+ax+bc=0$. [P. U. 1923.]

If a be the common root of the given equations, then we have

$$\left.\begin{aligned} a^2+ba+ca&=0\\ a^2+ca+ab&=0\end{aligned}\right\}$$

$$\therefore \quad \frac{a^2}{a(b^2-c^2)} = \frac{a}{a(c-b)} = \frac{1}{c-b}$$

or $\dfrac{a^2}{a(b+c)} = \dfrac{a}{-a} = \dfrac{1}{-1}$, b not being equal to c.

From here we see that the condition for a common root is $a+b+c=0$.....(1)
and that the common root is a.

Hence the other root of the first equation is $ca \div a = c$ and the other root of the second equation is $ab \div a = b$.

Now the equation whose roots are c and b is

$$x^2-(b+c)x+bc=0$$

or $$x^2+ax+bc=0. \quad [\text{Since } b+c=-a \text{ from (1)}]$$

Ex. 2. Find the condition that the quadratic expressions $lx^2+my+ny^2$ and $l'x^2+m'xy+n'y^2$ may have a common factor.

If $x-ay$ be the common factor, then suppose that

$$lx^2+mxy+ny^2=l(x-ay)(x+by).....(1)$$

and that $l'x^2+m'xy+n'y^2=l'(x-ay)(x+b'y).....(2)$

Since the right-hand side of (1) vanishes when $x=ay$, the left-hand side also must vanish for the same value of x.

$$\therefore \quad l(ay)^2+n(ay)y+ny^2=0$$

or $$la^2+ma+n=0.....(3)$$

Similarly from (2) $l'a^2+m'a+n'=0.....(4)$

From (3) and (4) $\dfrac{a^2}{mn'-m'n} = \dfrac{a}{nl'-n'l} = \dfrac{1}{lm'-l'm}$

whence the required condition is

$$(nl'-n'l)^2=(mn'-m'n)(lm'-l'm).$$

Aliter : The condition that the two expressions may have a common factor is the same as the condition that the corresponding equations may have a common root. etc. etc.

Exercise IV.

1. Form the quadratic equations whose roots are

(*i*) 3 and 4. (*ii*) 4 and -6.

(*iii*) $-\frac{2}{3}$ and 5. (*iv*) 18 and $\frac{2}{3}$.

(*v*) -7 and $-\frac{3}{4}$. (*vi*) $3+\sqrt{2}$ and $3-\sqrt{2}$.

(*vii*) $4+\sqrt{7}$ and $4-\sqrt{7}$. (*viii*) $2\pm i\sqrt{5}$.

(*ix*) $-6\pm i3$. (*x*) $\frac{a-b}{a+b}$ and $-\frac{a+b}{a-b}$.

2. If α, β be the roots of $3x^2+4x+5=0$, find the value of (*i*) $\alpha-\beta$, (*ii*) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$, (*iii*) $\alpha^3+\beta^3$.

3. If α, β be the roots of $lx^2+mx+n=0$, find the value of (*i*) $\alpha^2+\beta^2$ (*ii*) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ (*iii*) $\alpha^2\beta+\alpha\beta^2$.

4. If $\alpha\ \beta$ be the roots of $3x^2+5x-7=0$, find the value of (*i*) $\alpha^4+\alpha^2\beta^2+\beta^4$, (*ii*) $\alpha^2-\beta^2$ (*iii*) $(2\alpha^2+3)(2\beta^2+3)$.

5. If α, β be the roots of $x^2-px+q=0$, find the value of (*i*) $\alpha^4\beta+\alpha\beta^4$ (*ii*) $\frac{\alpha}{1+\beta}+\frac{\beta}{1+\alpha}$

(*iii*) $\frac{1-\alpha}{1+\alpha}+\frac{1-\beta}{1+\beta}$.

6. If α, β be the roots of $x^2-px+q=0$ and, Υ, δ those of $x^2-lx+m=0$, find the value of $(\alpha-\Upsilon)(\beta-\delta)+(\alpha-\delta)(\beta-\Upsilon)$.

7. If α, β be the roots of $px^2+qx+q=0$, show the $\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}+\sqrt{\frac{q}{p}}=0$.

8. If α, β be the roots of $x^2+px+q=0$, find the equation whose roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$.

9. If α, β be the roots of $x^2+lx-m=0$, form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

10. If α, β be the roots of $x^2+ax+b=0$, what equation will have the roots $\alpha^2+\alpha\beta$ and $\beta^2+\alpha\beta$?

11. If α, β be the roots of $ax^2+bx+c=0$, find the equation whose roots are *(i)* $\alpha^4\beta^7$ and $\alpha^7\beta^4$, *(ii)* $\alpha^2\beta^{-1}$ and $\alpha^{-1}\beta^2$, *(iii)* $\alpha^2\beta^{-2}$ and $\alpha^{-2}\beta^2$.

12. If α, β be the roots of $px^2+qx+r=0$, find the equation whose roots are $\frac{1}{a\alpha+b}$ and $\frac{1}{a\beta+b}$.

13. If α, β be the roots of $x^2+3x+5=0$, find the equation whose roots are $\alpha+2$ and $\beta+2$.

14. Find the equation whose roots are double the roots of $x^2-bx+c=0$.

15 Find the equation whose roots are the squares of the roots of $x^2-mx+n=0$.

16. Find the equation whose roots are k times the roots of the equation $ax^2+bx+c=0$.

17. Find the condition that one root of $lx^2+mx+n=0$ may be double of the other.

18 Find the condition that one root of $ax^2+bx+c=0$ may be four times the other.

19. If α, β be the roots of $x^2+x+1=0$, show that $\alpha^2=\beta$.

20. For what value of k will the equation $3x^2-(1+4k)x+k^2+5=0$ have a root one-third of the other.

21. Find the common root of the equations $lx^2+mx+n=0$ and $px^2+qx+r=0$

22. Show that the equations $ax^2+x+b=0$ and $bx^2+x+a=0$ will have a common root if $a=b$ or $a+b=1$.

23. For what values of p the equations $x^2-11x+24=0$ and $x^2-8x-p=0$ will have a common root ?

24. For what values of m the expressions $4x^2-35x+m$ and $3x^2-26x+16$ will have a common factor ?

25. Find the values of m for which the expressions $x^2+12xy+4my^2$ and $x^2+9xy+my^2$ will have a common factor.

26. If the equation $ax^2+bx+c=0$ is not changed when a, b and c are increased by k, show that $x^2+x+1=0$.

17. Nature of the roots of a Quadratic Equation.

If α, β be the roots of the equation $ax^2+bx+c=0$, then we have .

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Evidently the nature of α, β depends upon the value of the quantity under the radical *viz* $\sqrt{b^2-4ac}$ We, therefore, have the following cases :

I. If b^2-4ac be a positive quantity, then the value of $\sqrt{b^2-4ac}$ is real and therefore α, β are *real* and *unequal*.

II. If b^2-4ac be zero, then the value of $\sqrt{b^2-4ac}$ is zero, and therefore α, β are *real* and *equal*, or coincident. each of them being $\frac{-b}{2a}$.

III. If b^2-4ac be a negative quantity, then the value of $\sqrt{b^2-4ac}$ is imaginary, and therefore α, β are *imaginary* and *unequal* and *conjugate*.

IV. If b^2-4ac be a perfect square, then $\sqrt{b^2-4ac}$ is rational, and therefore α, β are *rational* and *unequal* otherwise irrational.

Cor. 1. The surd roots of an equation with rational coefficients occur in pairs. Thus if $l+\sqrt{m}$ be a root, then $l-\sqrt{m}$ is also a root, when $\sqrt{m}$ is a proper surd.

Cor. 2. The imaginary roots of an equation with rational coefficients occur in pairs. Thus if $p+iq$ be a root, then $p-iq$ is also a root, where q is a positive number.

Note. Since the expression b^2-4ac helps us to discriminate the nature of the roots, it is called the **Discriminant** of the equation $ax^2+bx+c=0$.

Ex. Examine the roots of the equations :—

(1) $2x^2-9x+8=0$, (2) $4x^2-12x+9=0$,

(3) $3x^2+4x+2=0$, (4) $2x^2+7x-4=0$.

(1) The quantity $b^2-4ac=81-4.2.8$
$=$a positive quantity which is not a perfect square.

The roots, therefore, are *real* and *unequal*, but *irrational*.

(2) The quantity $b^2-4ac=144-4.4.9$
$=0$.

The roots, therefore, are *real* and *equal*.

(3) The quantity $b^2-4ac=16-4.3.2$
$=$a negative quantity.

The roots, therefore, are *imaginary* and *unequal.*

(4) The quantity $b^2-4ac=+9+4.2.4$
$=81=9^2.$

The roots, therefore are *rational* and *unequal.*

18. Signs of the roots of the equation $ax^2+bx+c=0.$

Let α, β be the two roots, so that $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

(*i*) *a and b of the same sign, and* c *of the opposite sign.* Then, $\alpha+\beta$ is $-ve$
and $\alpha\beta$ is $-ve.$
$\therefore$ One root is $+ve$ and the geater of two is $-ve.$

Ex. $x^2+x-6=0.$ Here $x=-3, 2.$

(*ii*) *a and c of the same sign, and b of the opposite sign.*

Then, $\alpha+\beta$ is $+ve$
and $\alpha\beta$ is $+ve$
$\therefore$ α and β are both $+ve$

Ex. $x^2-5x+6=0.$ $x=3,$ $x=2.$

(*iii*) *b and c of the same sign, and a of the opposite sign.*

Then, $\alpha+\beta$ is $+ve$..... (1)
$\alpha\beta$ is $-ve$.....(2)

From (2), either α is $-ve$ or β is $-ve$ and from (1), the greater root must be $+ve.$

Ex. $x^2-x-6=0.$ Here roots are 3 and -2, the greater root being positive.

(*iv*) *a, b and c are all of the same sign.*

Then, $\alpha+\beta$ is $-ve.$
and $\alpha\beta$ is $+ve.$
$\therefore$ α and β are both $-ve.$

Ex. $x^2+5x+6=0.$ Here the roots -2 and -3 are both negative.

19. Quadratic Expressions. An expression of the form ax^2+bx+c is called a *quadratic expression.*

Evidently a quadratic expression equated to zero gives us a quadratic equation. A quadratic equation is satisfied by only two values of x. But in a quadratic expression we may give to x whatever value we like. In general a quadratic expression will assume different values for different values of x.

(*i*) We have already seen (Art. 12 Cor.) that if α, β be the roots of the equation $ax^2+bx+c=0$, then the factors of the expression ax^2+bx+c are $a(x-\alpha)(x-\beta)$. These factors are real and distinct, or imaginary according as α and β are real and different, equal or imaginary, *i. e.*, according as b^2-4ac is positive, zero or negative.

(*ii*) The expression ax^2+bx+c will have a factor $x-\lambda$ only when λ is a root of the equation $ax^2+bx+c=0$. For, if we actually divide ax^2+bx+c by $x-\lambda$ the remainder left is $a\lambda^2+b\lambda+c$. Now $x-\lambda$ will be a factor when there is no remainder *i. e.*, when $a\lambda^2+b\lambda+c=0$, or when $ax^2+bx+c=0$ is satisfied by $x=\lambda$

(*iii*) For all values of x, the expression ax^2+bx+c has the same sign as a, except when the roots of the equation $ax^2+bx+c=0$ are real and unequal and x lies between them. For, if α, β be the roots then

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

Now if x lies between α and β, one of the two factors will be negative and the expression will not be governed by the sign of a. But if x is greater or less than α and β both, then in either case the product of the two factors is positive and the sign of a governs that of the expression. If $\beta=\alpha$, then the expression $=a(x-\alpha)^2$. In this case x may be less than or greater than α, $(x-\alpha)^2$ is positive; and the sign of a is to be attached to the expression. Next, if α and β are imaginary, in which case b^2-4ac is $-ve$

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\},$$

So that b^2-4ac being negative, we have $4ac-b^2$ necessarily positive and $\left(x+\frac{b}{2a}\right)^2$ is positive; hence

$\left(x+\frac{b}{2a} \right)^2+\frac{4ac-b^2}{4a^2}$ is positive. Thus the expression ax^2+bx+c has the same sign as a.

Ex. Determine the sign of $3x^2+8x+11$

$$3x^2+8x+11=3\{x^2+\tfrac{8}{3}x+\tfrac{11}{3}\}$$
$$=3\{(x+\tfrac{4}{3})^2+\tfrac{17}{9}\}$$

Now the expression $(x+\frac{4}{3})^2+\frac{17}{9}$ is positive for all real values of x and therefore the given expression is positive.

20. Limiting values of Quadratic Expressions. We can, by the help of Arts 16 and 17, find the limiting values of a given quadratic expression The method will be clear from the following examples.

Ex. 1. For all real values of x find the limiting values of $\frac{x^2+2x-3}{x^2+2x-8}$

Suppose that y is a value of the given expression, so that $\frac{x^2+2x-3}{x^2+2x-8}=y$

or $$x^2(1-y)+2x(1-y)+(8y-3)=0.$$

Since x is real, the roots of this equation must be real

$\therefore$ Its discriminant $4(1-y)^2-4(1-y)(8y-3)$ must be positive

$\therefore$ $4(1-y)(4-9y)$ must be positive ;

i. e., both the factors should be either positive or both negative

Thus y must be greater than 1 or less than $\frac{4}{9}$.

Hence the limiting values of the given expression are 1 and $\frac{4}{9}$.

Ex. 2. If x be real, show that the expression $x^2-10x+27$ can never be *less* than 2.

Let $x^2-10x+27=m$ or $x^2-10x+27-m=0$

So that $x=5\pm\sqrt{m-2}$.

Now x will be real when $m-2$ is positive or zero, *i. e.* when m is greater than or at the most equal to 2. It means that m cannot be less than 2. Hence the expression can never be less than 2.

Ex. 3. In the equation $x^2 - px + q^2 = 0$ if x be real, show that p cannot lie between $\pm 2q$.

From the equation $x = \frac{1}{2}\{p \pm \sqrt{p^2 - 4q^2}\}$.

Now x will be real when $p^2 - 4q^2$ is positive or zero, *i. e.*, when $(p-2q)(p+2q)$ is positive or zero. For positive values of p and q we must have p greater than $2q$ or less than $-2q$; and for zero value we must have $p = \pm 2q$.
Hence p cannot lie between $\pm 2q$.

EXERCISE V.

1. Discuss the nature of the roots of the equations :—

(*i*) $3x^2 + 2x + 1 = 0$ (*ii*) $x^2 + 9x + 20 = 0$

(*iii*) $9x^2 + 5x - 1 = 0$ (*iv*) $4x^2 - 12x + 9 = 0$

(*v*) $3x^2 - x + 1 = 0$ (*vi*) $7x^2 - 13x + 4 = 0$

2. Show that the equation $3x^2 + 5x + 7 = 0$ cannot be satisfied by any real values of x.

3. For what values of p the equation $9x^2 - px + 25 = 0$ has equal roots ?

4. If the equation $4x^2 + 8x + m = 0$ has equal roots, find m.

5. Find the values of m if the equation $(1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ has equal roots.

6. Show that the roots of the equation $4x^2 - 8x + (4 - p^2 - q^2) = 0$ are real.

7. Prove that the equation $(a-b+c)x^2 + 4(a-b)x + (a-b-c) = 0$ has real roots.

8. Show that the equation $(a+c-b)x^2 + 2cx + (b+c-a) = 0$ has rational roots.

9. Show that the roots of $x^2 + px + q = 0$ will be rational if $p = k + \frac{q}{k}$, where p, q, k are all rational quantities.

10. Prove that $3x^2 - 5x + 3$ is always positive.

11. Show that $7x - 4 - 5x^2$ is always negative.

12. Show that $x^2 - 9x + 21$ cannot be less than $\frac{3}{4}$.

13. Prove that the least value of $5x^2 - 7x + 7$ is $\frac{3}{20}$.

14. Show that $3 + 5x - 2x^2$ cannot be greater than $6\frac{1}{8}$.

15. If x be real show that $\frac{x^2+14x+9}{x^2+2x+3}$ cannot lie between 4 and -5.

16. Prove that $\frac{x^2+34x-71}{x^2+2x-7}$ cannot take values between 5 and 9 and for all real values of x.

17. For real values of x prove that $x+\frac{1}{x}$ cannot lie between 2 and -2.

18. Show that $\frac{1}{x}(x^2-5x+9)$ must lie between 1 and -11 for real values of x.

19. For any real value of x the expression

$$\frac{1}{1+x}+\frac{1}{1+3x}-\frac{1}{(1+x)(1+3x)}$$

cannot lie between 1 and 4.

20. If p be greater than unity, then show that for all values of x the expression $\frac{x^2+2x+p^2}{x^2-2x+p^2}$ lies between $\frac{p+1}{p-1}$ and $\frac{p-1}{p+1}$.

21. Find the condition that the roots of $ax^2+bx+c=0$ may be (*i*) reciprocal of each other, (*ii*) both positive; (*iii*) both negative, (*iv*) one positive and the other negative (*v*) equal and opposite in sign.

22. Find the condition that the equation $ax^2+bx+c=0$ may have

(*i*) one root zero, (*ii*) both roots zero. (*iii*) one root infinite, (*iv*) both roots infinite.

23. Solve the equation $\frac{1}{x+p}-\frac{1}{x+r}=0$.

CHAPTER III

Simultaneous Quadratic Equations.

21. Simple Cases. Under this head will be treated such simultaneous equations of higher degree than the first as admit (*a*) the process of elimination of a variable involved ; or (*b*) the method of cross multiplication. These two methods being rather elementary are already known to the student.

Ex. 1. Solve the equations $\left.\begin{aligned} 3x-2y&=1 \\ x^2+y^2&=74 \end{aligned}\right\}$

From the first equation $y=\dfrac{3x-1}{2}$ .....(A)

Substituting this value of y in the second equation we have

$$x^2+\left(\frac{3x-1}{2}\right)^2=74,$$

or $$\frac{13}{4}x^2-\frac{3}{2}x-\frac{295}{4}=0,$$

or $$13x^2-6x-295=0,$$

whence $$x=\frac{3\pm\sqrt{9+13\times 295}}{13}=\frac{3\pm 62}{13}$$

$$=5,\ -5\tfrac{9}{13}.$$

Now if $x=5$, the relation (A) gives $y=7$; and if $x=-5\frac{9}{13}$, we get $y=-9\frac{5}{13}$.

Thus the solutions are $\left.\begin{aligned} x&=5,\ -5\tfrac{9}{13} \\ y&=7,\ -9\tfrac{5}{13} \end{aligned}\right\}$.

Ex 2. Solve $\left.\begin{aligned} 2x-3y&=11 \\ \frac{4}{x}-\frac{3}{y}&=-\frac{17}{7} \end{aligned}\right\}$.

From the first equation $y=\dfrac{2x-11}{3}$ .....(1)

With this value the second equation becomes

$$\frac{4}{x}-\frac{9}{2x-11}=-\frac{17}{7}$$

or $$17x^2-97x-154=0$$

whence $x=7$, or $-1\frac{2}{17}$.

Now when $x=7$, the relation (I) gives $y=1$, and when $x=-\frac{22}{7}$, then we get $y=-\frac{77}{17}$.

Hence the solutions are $\left.\begin{matrix} x=7, & -\frac{22}{7} \\ y=1, & -\frac{77}{17} \end{matrix}\right\}$.

Ex. 3. Solve $\left.\begin{matrix} x+y=12 \\ xy=32 \end{matrix}\right\}$.

First Method. From the first equation $y=12-x$

$\therefore$ the second equation gives $x^2-12x+32=0$,

whence $x=8$ or $x=4$.

Now when $x=8$, we get $y=4$;

and when $x=4$, we get $y=8$.

Second Method. We know that

$$(x-y)^2=(x+y)^2-4xy$$
$$=12^2-4\cdot 32=16.$$

$\therefore$ $x-y=\pm 4$.

Now the equation $x-y=4$ combined with $x+y=12$ gives $x=8$ and $y=4$;

and the equation $x-y=-4$ combined with $x+y=12$ gives

$x=4$ and $y=8$.

Hence the solutions are

$$\left.\begin{matrix} x=8, & 4 \\ y=4, & 8 \end{matrix}\right\}.$$

Ex. 4. Solve $\left.\begin{matrix} 5x^2+2y^2=5a^2+8b^2 \\ xy=2ab. \end{matrix}\right\}$ .....(1) .....(2)

We have $5x^2+2y^2=5a^2+8b^2$

and $2\sqrt{10}\,xy=4\sqrt{10}\,ab$

Whence by addition and subtraction we get two more equations, *viz.*,

$$(\sqrt{5}x+\sqrt{2}y)^2=(\sqrt{5}a+2\sqrt{2}b)^2$$

and $(\sqrt{5}x-\sqrt{2}y)^2=(\sqrt{5}a-2\sqrt{2}b)^2$

$\therefore$ $\sqrt{5}x+\sqrt{2}y=\pm(\sqrt{5}a+2\sqrt{2}b)$

and $\sqrt{5}x-\sqrt{2}y=\pm(\sqrt{5}a-2\sqrt{2}b)$

Hence by addition and subtraction we get

$$\left.\begin{matrix} x=a, & -a, & 2\sqrt{\frac{2}{5}}\,b, & -2\sqrt{\frac{2}{5}}\,b \\ y=2b, & -2b, & \sqrt{\frac{5}{2}}\,a, & -\sqrt{\frac{5}{2}}\,a \end{matrix}\right\}$$

Ex. 5. Solve $\left.\begin{matrix} 25x^2+9y^2=73 \\ 5x+3y=11 \end{matrix}\right\}$ .....(1) .....(2)

Subtracting (1) from the square of (2) we get

$$30xy=48,$$

which when subtracted from (1) gives

$$(5x-3y)^2=25$$

or $$5x-3y=\pm 5.$$

Combining $5x-3y=5$ with (2) we get

$$x=\tfrac{8}{5} \text{ and } y=1;$$

also combining $5x-3y=-5$ with (2) we get

$$x=\tfrac{3}{5} \text{ and } y=\tfrac{8}{5}.$$

Ex. 6. Solve $\left.\begin{aligned} 9x^2+5y^2-29=0 \\ 5x^2-3y^2+7=0 \end{aligned}\right\}$.

The method of cross multiplication gives

$$\frac{x^2}{35-87}=\frac{y^2}{-63-145}=\frac{1}{-27-25}$$

or $$\frac{x^2}{52}=\frac{y^2}{208}=\frac{1}{52}$$

whence $x^2=1$ and $y^2=4$

$\therefore$ $x=\pm 1$ and $y=\pm 2$.

Hence the solutions are

$$\left.\begin{aligned} x=1,\ \ 1,\ -1,\ -1 \\ y=2,\ -2,\ \ 2,\ -2 \end{aligned}\right\}$$

Ex. 7. Solve $\left.\begin{aligned} x^2+y^2=a^2 \\ y^2+z^2=b^2 \\ z^2+x^2=c^2 \end{aligned}\right\}$ ....(1) ..... (2) ..... (3)

Multiply (1) by b^2 and (2) by a^2 and subtract, then

$$b^2x^2+(b^2-a^2)y^2-a^2z^2=0 \text{ ;.....(4)}$$

also multiply (2) by c^2 and (3) by b^2 and subtract, then

$$-b^2x^2+c^2y^2+(c^2-b^2)z^2=0\text{.....(5)}$$

From (4) and (5) by cross multiplication

$$\frac{x^2}{(b^2-a^2)(c^2-b^2)+a^2c^2}=\frac{y^2}{a^2b^2-b^2(c^2-b^2)}$$

$$=\frac{z^2}{b^2c^2+b^2(b^2-a^2)}$$

or $$\frac{x^2}{a^2-b^2+c^2}=\frac{y^2}{b^2-c^2+a^2}=\frac{z^2}{c^2-a^2+b^2}$$

and each of these fractions

$$=\frac{x^2+y^2+z^2}{a^2+b^2+c^2}$$

$$=\frac{a^2+b^2+c^2}{2(a^2+b^2+c^2)}$$ [Since $2(x^2+y^2+z^2)=a^2+b^2+c^2$ from the given equations.

$$=\tfrac{1}{2}.$$

Hence $x^2=\frac{a^2-b^2+c^2}{2}$, $y^2=\frac{b^2-c^2+a^2}{2}$,

$$z^2=\frac{c^2-a^2+b^2}{2}.$$

Thus if we put $k_1=\sqrt{\frac{a^2-b^2+c^2}{2}}$,

$$k_2=\sqrt{\frac{b^2-c^2+a^2}{2}},\quad k_3=\sqrt{\frac{c^2-a^2+b^2}{2}},$$

the solutions are

$$\begin{array}{llllllll} x=k_1, & k_1, & k_1, & k_1, & -k_1, & -k_1, & -k_1, & -k_1 \\ y=k_2, & -k_2, & k_2, & -k_2, & -k_2, & -k_2, & k_2, & k_2 \\ z=k_3, & k_3, & -k_3, & -k_3, & -k_3, & k_3, & -k_3, & k_3. \end{array}$$

Ex. 8. Solve
$$\left.\begin{aligned} x+y+z&=a+b+c \\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}&=3 \\ x^2+y^2+z^2&=a^2+b^2+c^2 \end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots\ldots(1)\\ &\ldots\ldots\ldots\ldots(2)\\ &\ldots\ldots\ldots\ldots(3)\end{aligned}$$

From (1) we get $(x-a)+(y-b)+(z-c)=0$.....(4)

and from (2) we get $\frac{x-a}{a}+\frac{y-b}{b}+\frac{z-c}{c}=0$.....(5)

Now treating $x-a$, $y-b$, $z-c$ as our variables we have from (4) and (5)

$$\frac{x-a}{\frac{1}{c}-\frac{1}{b}}=\frac{y-b}{-\frac{1}{c}+\frac{1}{a}}=\frac{z-c}{\frac{1}{b}-\frac{1}{a}}$$

or $$\frac{x-a}{a(b-c)}=\frac{y-b}{b(c-a)}=\frac{z-c}{c(a-b)}$$

Putting each of these fractions equal to k, we get

$x=ka(b-c)+a$, $y=kb(c-a)+b$, $z=kc\,a-b)+c$

Substituting these value in (3),

$\{ka(b-c)+a\}^2+\{kb(c-a)+b\}^2+\{kc(a-b)+c\}^2=a^2+b^2+c^2.$

or $k^2[a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2]+2k[a(b-c)+b^2(c-a)+c^2(a-b)]=0,$

whence $k=0$,

or $$k=-2\,\frac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2}$$

$=R$ (Suppose).

Taking $k=0$, we have $x-a=0 \quad \therefore \quad x=a$
$y-b=0 \quad \therefore \quad y=b$
$z-c=0 \quad \therefore \quad z=c.$

Taking $k=\mathrm{R}$, we have $x-a=\mathrm{R}a(b-c) \quad \therefore x=\mathrm{R}a(b-c)+a$
$y-b=\mathrm{R}b(c-a) \quad \therefore y=\mathrm{R}b(c-a)+b$
$z-c=\mathrm{R}c(a-b) \quad \therefore z=\mathrm{R}c(a-b)+c.$

Exercise VI

Solve the following sets of equations :—

1. $3x^2-2y=23$
$4x-3y=6.$

2. $x^2+y^2=85$
$y-x=7.$

3. $2x^2+5y^2=130$
$x+y=9.$

4. $x^2-y^2=6$
$x+y=6.$

5. $2x^2+5y^2=22$
$3x+4y=11.$

6. $x^2+xy+5x+10y=29$
$x+2y=3.$

7. $3x-2y=11$
$\frac{3}{x}-\frac{4}{y}=-\frac{17}{7}.$

8. $2x+7y=11$
$\frac{3}{x}+\frac{2}{y}=\frac{7}{2}.$

9. $x+y=8$
$xy=15.$

10. $x-y=5$
$xy=14.$

11. $lx+my=3$
$lm\ xy=2.$

12. $x^2-y^2=9$
$xy=20.$

13. $2x^2+3y^2=24$
$xy=24.$

14. $4x^2+9y^2=793$
$2x-3y=-1.$

15. $x^2+y^2=13$
$x^2-y^2=5.$

16. $ax^2+by^2=a^3+b^3$
$bx^2-ay^2=ab(a-b).$

17. $2x^2-3y=20$
$x^2+5y=36.$

18. $7x+xy=20$
$2xy+5x=22.$

19. $x+6y+3z=0$
$3x+7y+z=0$
$x^2+y^2+z^2=102\frac{1}{2}.$

20. $x^2+y^2=13$
$x^2+z^2=34$
$y^2+z^2=29.$

21. $x(y+z)=5$
$y(x+z)=8$
$z(x+y)=9.$

22. $3x^2-2y^2+5z^2=0$
$7x^2-3y^2-15z^2=0$
$5x-4y+7z=6.$

23. $9x+y-8z=0$
$4x-8y+7z=0.$
$yz+zx+xy=47.$

24. $x^4+x^2y^2+y^4=133$
$x^2-xy+y^2=7.$

25. $x^2+xy+y^2=84$
$x-\sqrt{xy}+y=14.$

26. $x+\frac{1}{y}=5$
$y+\frac{1}{x}=\frac{5}{6}.$

27. $\frac{1}{x}+\frac{1}{y}=3$
$\frac{1}{xy}=2.$

28. $\frac{x}{y}-\frac{y}{x}=\frac{16}{15}$
$x-y=2.$

29. $\frac{1}{x}+\frac{1}{y}=\frac{1}{3}$
$x+y=16.$

30. $\frac{x^2}{y}+\frac{y^2}{x}=\frac{9}{2}$
$\frac{1}{x+y}=\frac{1}{3}.$

31. $\sqrt{x}+\sqrt{y}=5$
$xy=36.$

32. $\sqrt{x}+\sqrt{y}=7$
$x+y=37.$

33. $x^2y^2+4xy=96$
$x+y=6.$

34. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$
$x+y=10.$

35. $xy+y^2=16$
$x^2-y^2=32.$

22. Symmetrical Equations.—An equation which remains unaltered when the unknown numbers (x and y) are interchanged, is called a *symmetrical equation.*

Some examples of symmetrical equations were already included in Exercise VI ; now such equations will be solved by the method peculiar to them.

Ex. 1. Solve $\left.\begin{matrix}x^2+y^2=137\\x+y=15\end{matrix}\right\}$

Suppose that $x=u+v$, and $y=u-v$ so that from the second equation we get $(u+v)+(u-v)=15$, *i.e.*, $u=\frac{15}{2}$.

$\therefore \quad x=\frac{15}{2}+v$ and $y=\frac{15}{2}-v.$

Substituting these values of x and y in the first equation $(\frac{15}{2}+v)^2+(\frac{15}{2}-v)^2=137$

or $2v^2+2\cdot\frac{225}{4}=137$

or $4v^2=49$

$\therefore \quad v=\pm\frac{7}{2}.$

Now if $v=\frac{7}{2}$, we have $x=u+v=\frac{15}{2}+\frac{7}{2}=11$ and $y=u-v=\frac{15}{2}-\frac{7}{2}=4$; and if $v=-\frac{7}{2}$, we have $x=\frac{15}{2}-\frac{7}{2}=4$, and $y=\frac{15}{2}+\frac{7}{2}=11.$

Hence the solutions are $x=11,\ 4$ }
$y=4,\ 11.$ }

Ex. 2. $x^2+y^2-x-y=a$ } .....(1)
$xy+x+y=b$ } .....(2)

Adding together, $x^2+y^2+xy=a+b$

or $(x+y)^2=xy+a+b$

or $(b-xy)^2=xy+a+b$, since from (2)
$x+y=b-xy$

or $x^2y^2-(2b+1)xy=a+b-b^2$

giving $xy=\frac{1}{2}\{2b+1\pm\sqrt{(2b+1)^2-4(a+b-b^2)}$
$=\frac{1}{2}(2b+1\pm\sqrt{8b^2-4a+1})$
$=$ A and B (suppose).

Now when $xy=$A, we get from (2) $x+y=b-$A.

Put $x=u+v$ and $y=u-v$, so that $(u+v)+(u-v)$
$=b-$A, *i.e.*, $u=\dfrac{b-\text{A}}{2}$.

Thus $x=\dfrac{b-\text{A}}{2}+v$ and $y=\dfrac{b-\text{A}}{2}-v$.

Substituting these values in (1)

$$\left(\frac{b-\text{A}}{2}+v\right)^2+\left(\frac{b-\text{A}}{2}-v\right)^2-(b-\text{A})=a$$

or $2v^2+2\cdot\dfrac{(b-\text{A})^2}{4}=a+b-\text{A}$

or $4v^2=2(a+b-\text{A})-(b-\text{A})^2$.

From here we get two values of v which combined with $\dfrac{b-\text{A}}{2}$ will give two values of x and y each.

Again, taking $xy=$B we shall get two more values of x and y each by a similar process.

Hence there are four sets of values.

Ex. 3. Solve $(x+y)^2-(x+y)=4xy$ } (1)
$2(x+y)^2(x^2+y^2)=16x^2y^2+11(x+y)^2$ } (2)

(2) May be written as

$2(x+y)^2\{(x+y)^2-2xy\}=16x^2y^2+11(x+y)^2$

or $2(x+y)^4-4xy(x+y)^2=16x^2y^2+11(x+y)^2$

Substituting for $4xy$ and $16x^2y^2$ from (1) in this

$2(x+y)^4-(x+y)^2\{(x+y)^2-(x+y)\}=(x+y)^4-2(x+y)^3$
$+(x+y)^2+11(x+y)^2$

whence $(x+y)^2=0$.....(3)

or $$(x+y)^2+(x+y)=(x+y)^2-2(x+y)+12$$

or $$x+y=4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now put $x=u+v$ and $y=u-v$, so that from (4) $u=2$.

Thus writing $2+v$ for x and $2-u$ for y in (1)

$$16-4=4(2+v)(2-v)$$

or $$3=4-v^2$$

or $$v=\pm 1$$

Taking $v=1$ we get $x=3$ and $y=1$

Taking $v=-1$ we get $x=1$ and $y=3$

Using (3) with (1) we get $x=0$ and $y=0$

Hence the solutions are $x=3,\ 1,\ 0$ }
$y=1,\ 3,\ 0$ } .

Ex. 4. Solve $\left.\begin{aligned}\frac{bx}{y+b}+\frac{ay}{x+a}&=\frac{a+b}{2}\\ \frac{x}{a}+\frac{y}{b}&=2\end{aligned}\right\}$ ……(1) ……(2)

Let $\frac{x}{a}=u+v$ and $\frac{y}{b}=u-v$, so that from (2) $u=1$

Now (1) may be written as

$$\frac{x}{\frac{y}{b}+1}+\frac{y}{\frac{x}{a}+1}=\frac{a+b}{2},$$

which on substituting $1+v$ for $\frac{x}{a}$ and $1-v$ for $\frac{y}{b}$ gives

$$\frac{a(1+v)}{2-v}+\frac{b(1-v)}{2+v}=\frac{a+b}{2}$$

or $$a\left\{\frac{1+v}{2-v}-\frac{1}{2}\right\}+b\left\{\frac{1-v}{2+v}-\frac{1}{2}\right\}=0$$

or $$\frac{a}{2}\cdot\frac{3v}{2-v}+\frac{b}{2}\cdot\frac{-3v}{2+v}=0.$$

Thus $v=0$ or $\frac{a}{2-v}=\frac{b}{2+v}$ giving $v=\frac{2(b-a)}{a+b}$

Taking $v=0$ we get $x=a$ and $y=b$.

Taking $v=\frac{2(b-a)}{a+b}$ we get

$$\frac{x}{a}=1+\frac{2(b-a)}{a+b}.$$

$$\therefore \quad x=\frac{a(3b-a)}{a+b}.$$

and $\frac{y}{b}=1-\frac{2(b-a)}{(a+b)}$,

$\therefore \quad y=\frac{b(3a-b)}{a+b}$.

Hence the solutions are $x=a, \frac{a(3b-a)}{a+b}$ }

$y=b, \frac{b(3a-b)}{a+b}$ }.

Exercise VII.

Solve the equations :—

1. $xy=32$
$x+y=12$.

2. $x^2+y^2=61$
$x+y=11$.

3. $xy=\frac{1}{2}$
$\frac{1}{x}+\frac{1}{y}=3$.

4. $x^2+y^2=181$
$xy=-90$.

5. $x^2+y^2+x+y=18$
$xy=6$.

6. $x^2+y^2-x+y=20$
$xy+x+y=1$.

7. $x+y+\sqrt{x+y}=12$
$x^3+y^3=189$.

8. $xy(x+y)=84$
$xy+x+y=19$.

9. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$
$x+y=10$.

10. $x^2+3xy=45$
$xy+4y^2=76$.

11. $x^2+y^2+5xy-2(x+y)=123$
$3xy-4(x+y)=24$.

12. $\frac{x}{a}+\frac{y}{b}=\frac{b}{a}+\frac{a}{b}$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{b^2}{a^2}+\frac{a^2}{b^2}$.

13. $\frac{a}{x}+\frac{b}{y}=4$
$\frac{x}{a}+\frac{y}{b}=1$.

14. $\frac{a}{a+x}+\frac{b}{b+y}=1$
$x+y=a+b$.

15. $\sqrt{ax}+\sqrt{by}=a+b$
$x+y=2(a+b)$.

23. Homogeneous Equations :—The equations in which all the terms containing the unknown quantities are of the same degree, are called the *homogeneous equations*. Such equations can conveniently be solved as illustrated below :—

Ex. 1. Solve $\left.\begin{aligned} x^2-2xy+3y^2&=9 \\ x^2-4xy+5y^2&=5 \end{aligned}\right\}$ .....(1) .....(2)

Let $y=vx$, so that (1) bscomes $x^2(1-2v+3v^2)=9$...(3)

and (2) becomes $x^2(1-4v+5v^2)=5$...(4)

Dividing (3) by (4) $\dfrac{1-2v+3v^2}{1-+v+5v^2}=\dfrac{9}{5}$

or $15v^2-13v+2=0$

whence $v=\frac{1}{5}$ or $\frac{2}{3}$.

Taking $v=\frac{1}{5}$ and substituting in (3) we have

$$x^2=\frac{9}{1-\frac{2}{5}+\frac{3}{25}}$$

$$=\frac{25}{9}$$

$$\therefore \quad x=\pm\frac{5}{\sqrt{2}}.$$

and hence $y=vx=\frac{1}{5}\times(\pm\frac{5}{\sqrt{2}})=\pm\frac{1}{\sqrt{2}}$

Taking $v=\frac{2}{3}$ and substituting in (3) we get

$$x^2=\frac{9}{1-\frac{4}{3}+\frac{12}{9}}=9$$

$$\therefore \quad x=\pm 3$$

and hence $y=vx=\frac{2}{3}\times(\pm 3)$

$$=\pm 2.$$

Thus the solutions are $x=3,\ -3,\ \dfrac{5}{\sqrt{2}}$, and $-\dfrac{5}{\sqrt{2}}$

$$y=2,\ -2,\ \frac{1}{\sqrt{2}},\text{ and } -\frac{1}{\sqrt{2}}$$

Ex. 2. Solve $\left.\begin{aligned} x^2-3xy+2y^2&=0 \\ 2x^2-3x+y^2&=24 \end{aligned}\right\}$ .....(1) .....(2)

Notice here both equations are not homogeneous, yet the solution can be effected by the same substitution as in Ex. 1. because of the right-hand side member of (1) being zero.

Put $y=vx$, so that (1) gives $1-3v+2v^2=0$

whence $v=1$ or $\frac{1}{2}$

$$\therefore \quad y=x \text{ or } \frac{x}{2}.$$

Taking $y=x$, (2) gives

$$2x^2-3x+x^2=24$$

or $x^2-x-8=0$

$\therefore \quad x = \frac{1}{3}(1 \pm \sqrt{33})$

Then also $y = \frac{1}{3}(1 \pm \sqrt{33})$.

Taking $y = \frac{x}{2}$, (2) gives

$$2x^2 - 3x + \frac{x^2}{4} = 24$$

or $3x^2 - 4x - 32 = 0$

$\therefore \quad x = 4, \text{ or } -\frac{8}{3}$

Then $y = 2, \text{ or } -\frac{4}{3}$.

Hence the solutions are $x = 4, \frac{-8}{3}$, and $\frac{1}{2}(1 \pm \sqrt{33})$

$y = 2, -\frac{4}{3}$, and $\frac{1}{2}(1 \pm \sqrt{33})$.

Exercise VIII

Solve the following equations :—

1. $x^2 + xy = 78$
$y^2 - xy = 7$.

2. $x^2 - xy + y^2 = 21$
$y^2 - 2xy = -15$.

3. $x^2 + y^2 + 3xy = 61$
$x^2 - y^2 + 2xy = 31$.

4. $y^2 + 2xy = 40$
$x^2 + 3xy = 45$.

5. $x^2 + y^2 - xy = 7$
$4x^2 - 120xy = -324$.

6. $5x^2 + 2y^2 = 5a^2 + 8b^2$
$xy = 2ab$.

7. $\frac{x^2 + y^2}{x^2 y^2} = \frac{1}{b^2}$
$\frac{x + y}{xy} = \frac{1}{a}$.

8. $\frac{x^2 + y^2}{x^2 - y^2} = \frac{13}{5}$
$2x^2 - 3y^2 = 24$.

9. $3x^2 + xy + y^2 = 15$
$31xy - 3x^2 - 5y^2 = 45$.

10. $4x^2 + 7y^2 = 148$
$12(x^2 + y^2) = 25xy$.

24. Miscellaneous. Generally the nature of the given equations suggests the method of procedure.

Ex. 1. Solve $\left(1 - \frac{2y}{x+y}\right)^2 + \left(1 + \frac{2y}{x-y}\right)^2 = \frac{82}{9}$...(1)

$3x + 7y = 26$....(2)

(1) may be written as $\left(\frac{x-y}{x+y}\right)^2 + \left(\frac{x+y}{x-y}\right)^2 = \frac{82}{9}$

which on writing t for $\frac{x-y}{x+y}$ becomes $t^2 + \frac{1}{t^2} = \frac{82}{9}$

or $9t^4 - 82t^2 + 9 = 0$

$\therefore \quad t = \pm 3, \pm \frac{1}{3}$

(*i*) If $t=3$, we have $\frac{x-y}{x+y}=3$

Componendo and dividendo gives $\frac{x}{y}=-\frac{4}{2}=-2$

or $x=-2y.$

and $\therefore$ (2) gives $x=-52$ and $y=26.$

(*ii*) Taking $t=-3$, we have $\frac{x-y}{x+y}=-3.$

or $\frac{x}{y}=\frac{-2}{4}=-\frac{1}{2}$

$\therefore$ $2x=-y$

and $\therefore$ (2) gives $x=-\frac{26}{11}$, $y=\frac{52}{11}$.

(*iii*) Taking $t=\frac{1}{3}$, we have $\frac{x-y}{x+y}=\frac{1}{3}$.

or $\frac{x}{y}=\frac{4}{2}=2$

$\therefore$ $x=2y$

and $\therefore$ (2) gives $x=4$, $y=2.$

(*iv*) Taking $t=-\frac{1}{3}$, we have $\frac{x-y}{x+y}=-\frac{1}{3}$.

giving $2x=y$

and $\therefore$ from (2) $x=-\frac{26}{17}$, $y=\frac{52}{17}$.

Hence $x=-52,\ 4,\ -\frac{26}{11}$, and $\frac{26}{17}$

$y=\ 26,\ 2,\ \frac{52}{11}$, and $\frac{52}{17}$.

Ex. 2. Solve $\left.\begin{aligned}\frac{x^3+y^3}{(x+y)^2}+\frac{x^3-y^3}{(x-y)^2}&=\frac{43}{8}x\\ 5x-7y&=4\end{aligned}\right\}\ \begin{aligned}&\ldots\ldots\ldots(1)\\&\ldots\ldots\ldots\ldots(2)\end{aligned}$

(1) may be written as

$$\frac{(x+y)^3-3xy(x+y)}{(x+y)^2}+\frac{(x-y)^3+3xy(x-y)}{(x-y)^2}=\frac{43}{8}x$$

or $$x+y-\frac{3xy}{x+y}+x-y+\frac{3xy}{x-y}=\frac{43}{8}x$$

or $$3xy\left(\frac{1}{x-y}-\frac{1}{x+y}\right)=\frac{43}{8}x-2x$$

$$=\frac{27}{8}x,$$

giving $x=0$ or $y\frac{x+y-x+y}{x^2-y^2}=\frac{9}{8}$

i.e., $$\frac{2y^2}{x^2-y^2}=\frac{9}{8}$$

i.e., $$9x^2=25y^2$$

i.e., $$3x=\pm 5y.$$

(*i*) Taking $x=0$, we get from (2), $y=-\frac{4}{7}$.

(*ii*) Taking $3x=5y$ we get from (2) $x=5$, $y=3$.

(*iit*) Taking $3x=-5y$ we get from (2) $x=\frac{10}{53}$, $y=-\frac{6}{53}$.

Ex. 3. Solve $\left.\begin{aligned}15xy+10x-9y&=6\\8xy-12x+10y&=15\end{aligned}\right\}$ .....(1) .....(2)

(1) may be written as $(5x-3)(3y+2)=0$,

whence $5x=3$ or $3y=-2$.

Also (2) may be written as $(4x+5)(2y-3)=0$.

giving $4x=-5$ or $2y=3$.

Combining $5x=3$ with $2y=3$ we get $x=\frac{3}{5}$, $y=\frac{3}{2}$;

and combining $3y=-2$ with $4x=-5$, we get

$$x=-\tfrac{5}{3},\quad y=-\tfrac{2}{3}.$$

Ex. 4. Solve $\left.\begin{aligned}x+3(x+y)^{\frac{1}{2}}&=18-y\\x^2-y^2+6y&=9.\end{aligned}\right\}$ .....(1) .....(2)

(1) may be written as $(x+y)+3(x+y)^{\frac{1}{2}}-18=0$.

or $\{(x+y)^{\frac{1}{2}}+6\}\{(x+y)^{\frac{1}{2}}-3\}=0$,

giving $x+y=36$ or 9.

Also from (2) $x^2-y^2+6y-9=0$.

or $(x+y-3)(x-y+3)=0$.

giving $x+y=3$ or $x-y=-3$.

Thus we have four sets of simple simultaneous equations.

(*i*) $x+y=36$, $x+y=3$. or (*ii*) $x+y=9$, $x+y=3$.

(*iii*) $x+y=36$, $x-y=-3$. or (*iv*) $x+y=9$, $x-y=-3$.

The sets (*i*) and (*ii*) are absurd, and the sets (*iii*) and (*iv*) give the solutions $\left.\begin{aligned}x&=\tfrac{33}{2},\ 3\\y&=\tfrac{39}{2},\ 6\end{aligned}\right\}$.

Ex. 5. Solve $\left.\begin{aligned}x(x+y+z)&=a\\y(x+y+z)&=b\\z(x+y+z)&=c\end{aligned}\right\}$.

Adding together $(x+y+z)(x+y+z)=a+b+c$.

$\therefore\quad x+y+z=\pm\sqrt{a+b+c}$.

Dividing this result by each equation in succession, we get

$$x=\pm\frac{a}{\sqrt{a+b+c}}$$

$$y=\pm\frac{b}{\sqrt{a+b+c}}$$

$$z=\pm\frac{c}{\sqrt{a+b+c}}.$$

Ex. 6. Solve $\left.\begin{aligned}x(y+z)&=l\\y(z+x)&=m\\z(x+y)&=n\end{aligned}\right\}$.

Adding together $2(xy+yz+zx)=l+m+n.$

or $$xy+yz+zx=\frac{l+m+n}{2}.$$

Subtracting from this result each equation in succession.

$$yz=\frac{m+n-l}{2},\quad zx=\frac{n+l-m}{2},\quad xy=\frac{l+m-n}{2}\ldots\ldots(A)$$

Multiply the results (A) together,

$$\therefore\quad x^2y^2z^2=\frac{(m+n-l)(n+l-m)(l+m-n)}{8}$$

or $$xyz=\pm\frac{1}{2}\sqrt{\frac{(m+n-l)(n+l-m)(l+m-n)}{2}}.$$

Now dividing this by each of the results (A) we get

$$x=\pm\sqrt{\frac{(l+m-n)(n+l-m)}{2(n+m-l)}},$$

$$y=\pm\sqrt{\frac{(l+m-n)(m+n-l)}{2(n+l-m)}},$$

$$z=\pm\sqrt{\frac{(m+n-l)(n+l-m)}{2(l+m-n)}}.$$

Ex. 7. Solve $xyz=\frac{y^2+z^2}{a}=\frac{z^2+x^2}{b}=\frac{x^2+y^2}{c}$.

From the equations $\left.\begin{aligned}\frac{x}{zx}+\frac{z}{xy}&=a\\ \frac{z}{xy}+\frac{x}{yz}&=b\\ \frac{x}{yz}+\frac{y}{zx}&=c\end{aligned}\right\}$ .....(A)

$\therefore$ by adding together $2\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right)=a+b+c,$

and subtracting from this twice of each of (A) we get

$$\left.\begin{aligned}\frac{2x}{yz}&=b+c-a\\ \frac{2y}{zx}&=c+a-b\\ \frac{2z}{xy}&=a+b-c\end{aligned}\right\} \text{ ....(B)}$$

Multiplying the results (B) two by two, we get

$$\frac{4}{x^2}=(c+a-b)(a+b-c)$$

$$\frac{4}{y^2}=(a+b-c)(b+c-a)$$

$$\frac{4}{z^2}=(b+c-a)(c+a-b)$$

whence $x=\pm\frac{2}{\sqrt{(c+a-b)(a+b-c)}}$,

$$y=\pm\frac{2}{\sqrt{(a+b-c)(b+c-a)}},$$

$$z=\pm\frac{2}{\sqrt{(b+c-a)(c+a-b)}}.$$

Ex. 8. Solve $\left.\begin{aligned}(x+y)(x+z)&=40\\ (y+z)(y+x)&=35\\ (z+x)(z+y)&=56\end{aligned}\right\}$.

Multiplying together

$$(x+y)^2(y+z)^2(z+x)^2=40\times35\times56$$

$\therefore$ $(x+y)(y+z)(z+x)=\pm280.$

Dividing this result by each given equation, we get

$$\left.\begin{aligned}y+z&=\pm7\\ z+x&=\pm8\\ x+y&=\pm5\end{aligned}\right\}.$$

These two sets of simple equations give
$x=\pm 3,\quad y=\pm 2,\quad z=\pm 5.$

EXERCISE IX.

Solve for the variables:

1. $x^2+y^2+2x-3y=35$
$x^2-y^2+2x+3y=35$

2. $3x^2+5y^2+4x+3y=9$
$3x^2+5y^2+2x-4y=14$

3. $x^2+y^2+x-y=5a^2+a$
$(x^2+y^2)(x-y)=5a^3.$

4. $2x+3y+6xy=11$
$4x^2+9y^2+12xy=36x^2y^2-11$

5. $xy=133-x^2-y^2$
$\sqrt{xy}=19-x-y.$

6. $xy=24-x-x^2$
$=32-y-y^2.$

7. $(x-2)^2+(y+2)^2=17$
$(x-1)^2+(y+1)^2=13.$

8. $xy=x+y$
$=x^2-y^2$

9. $(9x+y)(x+y)=273$
$(9x-y)(x-y)=33$

10. $2(x^2+y^2)-6xy+3(x+y)=44$
$x^2+y^2-4xy+5(x+y)=32$

11. $\sqrt{x^2-y^2}+x=8$
$x=1+y$

12. $xy+x+y=11$
$xy(x+y)=30$

13. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}-\frac{7}{\sqrt{xy}}=1$
$\sqrt{xy}(x+y)=78$

14. $3xy-4(x+y)=0$
$x^2+y^2+x+y=26$

15. $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}=\frac{27}{4}$
$x-y=2$

16. $\sqrt{x+y}+\sqrt{x-y}=4$
$x^2-y^2=9$

17. $6x+5y-\frac{6}{x}-\frac{5}{y}=29\frac{1}{3}$
$3x+4y-\frac{3}{x}-\frac{4}{y}=18\frac{2}{3}$

18. $2x^2-xy+y^2=2y$
$2x^2+4xy=5y$

19. $\frac{x-y}{x+y}-\frac{2x+y}{3x-y}=-2\frac{8}{15}$
$7x+5y=29$

20. $x^2+a^2=y^2+b^2=(x+y)^2+(a-b)^2$

21. $x^2+y^2=\dfrac{26x}{3y}$

$x^2-y^2=\dfrac{15y}{2x}$

22. $(y-z)(z+x)=22$
$(z+x)(x-y)=33$
$(x-y)(y-z)=6$

23. $xy=x+y$
$zx=2(z+x)$
$yz=3(y+z)$

24. $x^2+xy+xz=45$
$y^2+yz+yx=75$
$z^2+zx+zy=105$

25. $2yz+x^2-y^2-z^2=3$
$2zx-x^2+y^2-z^2=-9$
$2xy-x^2-y^2+z^2=-3$

Miscellaneous Questions I.

1. Solve the equations :—

(*i*) $15x^{-\frac{1}{n}}-8+x^{\frac{1}{n}}=0$ (*ii*) $2^{2x+3}-32\,2^x+1=0.$

(*iii*) $(4x^2-9x+1)^2+7(4x^2-9x+1)+6=0,$

(*iv*) $\sqrt{x^2-8x+15}+\sqrt{x^2+2x-15}=\sqrt{4x^2-18x+15}$

2. Find the limits between which $\dfrac{x^2-2x-3}{2x^2+2x+1}$ does not lie for real values of x.

3. If α, β be the roots of $x^2+4x+3=0$ show that $1+\dfrac{\beta}{\alpha}$ and $1+\dfrac{\alpha}{\beta}$ are the roots of $3x^2-16x+16=0$.

4. Find the least value of $x+\dfrac{9}{x}$ when x is real.

5. Evaluate $(2+3\sqrt{-5})^{\frac{1}{3}}+(2-3\sqrt{-5})^{\frac{1}{3}}$

6. Show that the roots of $(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0$ are real and that they cannot be equal unless $a=b=c$.

7. If the equation $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{c}{x+c}+\dfrac{d}{x+d}$ has a pair of equal roots, show that either one of the quantities a, b equals to c, d or else

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}.$$

8. Show that the least value of $x+\frac{1}{x-2}$ is 4.

9. If the equations $x^2-7x+a=0$ and $x^2-9x+2a=0$ have a common root, show that it is 0 or 5

10. If the equations $ax^2+bx+c=0$ and $a'x^2-b'x+c'=0$ have one root the same but of opposite sign, show that the other roots satisfy the equation

$$\frac{x^2}{\frac{b}{a}+\frac{b'}{a'}}+x+\frac{1}{\frac{b}{c}+\frac{b'}{c'}}=0.$$

[Let α, β and $-\alpha$, Υ be the roots. Then $\alpha+\beta=\frac{-b}{a}$, $\alpha\beta=\frac{c}{a}$, $\Upsilon-\alpha=-\frac{b'}{a'}$, $\Upsilon\alpha=-\frac{c'}{a'}$. Now find the values of $\beta+\Upsilon$ and $\beta\Upsilon$.]

11. If the equations $ax^2+bx+c=0$ and $px^2+qx+r=0$ have a common root, and if the other root of the second be the square of the other root of the first, show that $abcpr+a^3r^2+c^3p^2=0$.

[Let α, β and α, β^2 be the roots, then $\alpha+\beta=\frac{-b}{a}$, $\alpha\beta=\frac{c}{a}$, $\alpha\beta^2=\frac{r}{p}$. Eliminate α and β.]

12. Discuss the nature of the roots of the equation $ax^2-2bx+c=0$

13. If one root of $ax^2+bx+c=0$ be the square of the other, find c when $a=8$, $b=-30$.

[P. U. 1931],

14. If α, β be the roots of the equation $x^2+lx+m=0$ find the value of

i) $(1+\alpha+\alpha^2)(1+\beta+\beta^2)$ and *ii*) $(w\alpha+w^2\beta)(w^2\alpha+w\beta)$ where w is a cube root of unity.

15. Solve the equation

$$\sqrt{x^2-x}+\sqrt{x-2}+\sqrt{x^2+3x^2+3x+2}=\sqrt{2}\sqrt{x^3+2x^2+2x+1}$$

16. If $\begin{matrix}3x+2yi\\5i-2\end{matrix}=\begin{matrix}15\\8x+3yi\end{matrix}$ find x and y.

17. If α, β be the roots of $x^2-2ax+b^2=0$ and α', β' those of $x^2-2cx+d^2=0$ and if $\alpha\alpha'+\beta\beta'=4p^2$ prove that $(a^2-b^2)(c^2-d^2)=(ac-2p^2)^2$.

[Evaluate $(\alpha+\beta)(\alpha'+\beta')+(\alpha-\beta)(\alpha'-\beta')$ and substitute].

18. If $\alpha\pm\sqrt{\beta}$ be the roots of $x^2+px+q=0$, show that $\frac{1}{\alpha}\pm\frac{1}{\sqrt{\beta}}$ are the roots of $(p^2-4q)(p^2x^2+4px)=16q$.

19. Find the quadratic equation whose sum of the roots is a and the sum of their cubes is b^3.

20. If $ar>bp$ show that the greatest value of $\frac{ax+b}{(px+r)^2}$ is $\frac{a^2}{4p(ar-bp)}$

21. Solve the equations :—

(*i*) $x^2y^2+5xy=84$
$x-y=6$

(*ii*) $x+y+3(x+y)^{\frac{1}{2}}=18$
$x^2-y^2=9$
} [Solve the first equation as quadratic in $\sqrt{x+y}$.]

(*iii*) $x^2+4y^2-15x=10(3y-8)$
$xy=6$
} [Make the first equation a quadratic in $x+2y$.]

(*iv*) $xyz=231$
$xyw=420$
$xzw=660$
$yzw=1540$

22. Show that the equations $ax^2+bx+c=0$ and $cx^2-bx+a=0$ are such that the roots of one are the negative reciprocal roots of the other.

23. For what value of λ will the equation

$$\frac{6}{x+1}-\frac{\lambda}{x}+\frac{10}{x-3}=0$$

have a pair of equal roots ?

24. Show that the roots of the equation $a^2x^2+(2ac-b^2)x+c^2=0$ are squares of the roots of the equation $ax^2+bx+c=0$.

25. If the expression $\frac{7x^2+14x-9}{2x^2+x-3}$ is less than unity, then $\frac{2}{5} > x > -3$.

26. Solve the equation

$$\frac{1}{x+4} = \frac{1}{x+5}.$$

CHAPTER IV

Partial Fractions.

25. The student is already familiar with the process of compounding or simplifying a given number of fractions connected by the signs of addition and subtraction into one. We shall now consider the converse process of decomposing a given fraction into simpler ones. These simpler fractions are called **Partial Fractions.**

For instance $\frac{2}{x-2}+\frac{1}{x+3}=\frac{3x+4}{x^2+x-6}$. Here $\frac{2}{x-2}$ and $\frac{1}{x+3}$ are the partial fractions of $\frac{3x+4}{x^2+x-6}$.

We know that the simplified form of a given number of fractions has for its denominator the L. C. M. of the denominators of the given fractions. Conversely, therefore, the partial fractions will have denominators whose L. C. M. is the denominator of the given fraction. Our first and foremost effort will, therefore, be to find all possible factors having for their L. C. M. the denominator of the given fractions.

According to the factors of the denominator four different cases may arise :—

I. All the factors may be linear and unrepeated.

II. All the factors may be linear, one or more of them being repeated.

III. There may be quadratic or higher degree factors unrepeated.

IV. One or more of these quadratic or higher degree factors may be repeated.

We shall take these cases one by one and illustrate the process of decomposition into partial fractions by examples.

It should be remembered that by *a proper fraction* or more simply *a fraction* we mean only such a fraction whose numerator is less than the denominator. If the degree of the numerator is greater than or equal to that of the denominator we can by ordinary division split a

given fraction into the form : Quotient $+\frac{\text{remainder}}{\text{divisor}}$ where the degree of the remainder is less than that of the divisor.

Ex. 1. $\frac{x^3+3x^2-x-4}{x^2+x+6}$ is not a proper fraction for the degree of the numerator is greater than that of the denominator. By actual division, it $=x+2-\frac{9x+14}{x^2+x+6}$.

Here the part $\frac{9x+14}{x^2+x+6}$ is a proper fraction.

Ex. 2. $\frac{x^2+7x+6}{x^2+4x-45}$ is not a proper fraction since the degree of the numerator and denominator is the same. By division it $=1+\frac{3(x+17)}{x^2+4x-45}$. Here $\frac{3(x+17)}{x^2+4x-45}$ is a proper fraction.

26. Case I. The denominator having linear but a non-repeated factors.

Ex. 1. Split $\frac{3x+4}{x^2+x-6}$ into partial fractions.

We have $\frac{3x+4}{x^2+x-6}=\frac{3x+4}{(x+3)(x-2)}$.

Let $\frac{3x+4}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$.

Then $\frac{3x+4}{(x+3)(x-2)}\equiv\frac{A(x-2)+B(x+3)}{(x+3)(x-2)}$.

$\therefore \quad 3x+4\equiv A(x-2)+B(x+3)$.

i.e., $3x+4\equiv x(A+B)+(3B-2A)$. $\quad (i)$

In this indentity, which is true for all values of x, put $x=-3$ and $x=2$. When $x=-3$, we get $A=1$ and when $x=2$, we get $B=2$.

Hence

$$\frac{3x+4}{x^2+x-6}=\frac{1}{x+3}+\frac{2}{x-2}.$$

Aliter : From identity (i) we have by Art. 11, on equating the coefficients of like powers of x,

$$3 = A + B$$
$$4 = -2A + 3B.$$

Whence, $A = 1$, $B = 2$

Ex. 2. Put into partial fractions

$$\frac{6x^3 + 18x^2 + 10x}{(2x+1)(x+2)(3x+2)}.$$

Here since the numerator is not of lower degree than the denominator, we have

$$\frac{6x^3 + 20x^2 + 10x}{(2x+1)(x+2)(3x+2)} = \frac{6x^3 + 20x^2 + 10x}{6x^3 + 19x^2 + 16x + 4}$$
$$= 1 + \frac{x^2 - 6x - 4}{(2x+1)(x+2)(3x+2)}.$$

We shall now break $\frac{x^2 - 6x - 4}{(2x+1)(x+2)(3x+2)}$ into partial fractions.

Let $\frac{x^2 - 6x - 4}{(2x+1)(x+2)(3x+2)} \equiv \frac{A}{2x+1} + \frac{B}{x+2} + \frac{C}{3x+2}$

$$\equiv \frac{A(x+2)(3x+2) + B(2x+1)(3x+2) + C(2x+1)(x+2)}{(2x+1)(x+2)(3x+2)}$$

$\therefore \quad x^2 - 6x - 4 \equiv A(x+2)(3x+2) + B(2x+1)(3x+2) + C(2x+1)(x+2).$

From this indentity when $x = -\frac{1}{2}$, we get $A = -1$; when $x = -2$, we get $B = 1$; and when $x = -\frac{2}{3}$, we get $C = -1$.

Hence the given fraction

$$= 1 + \frac{1}{2x+1} + \frac{1}{x+2} - \frac{1}{3x+2}.$$

Note.—To find the numerator of a partial fraction, whose denominator is a factor of the given fraction, substitute the value got by equating the corresponding denominator to zero, in the identity except in the factor which is the denominator of the partial fraction. The value so obtained is the required numerator.

Thus, let $\frac{R(x)}{F(x)}$ be a proper fraction and let $x - a$ be a factor of $F(x)$.

Then $\frac{R(x)}{F(x)} = \frac{R(x)}{(x-a)f(x)}$ where $F(x) = (x-a)f(x)$

Assuming $\frac{R(x)}{F(x)} = \frac{R(x)}{(x-a)f(x)} = \frac{A}{x-a} + \frac{f_1(x)}{f(x)}$,

the value of A is $\frac{R(a)}{F(a)}$

Example : $\frac{2}{x^2-5x+6} = \frac{2}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

The value of $A = \frac{2}{3-2} = 2$

and .. .. .. $B = \frac{2}{-1} = -2$

Hence etc. etc.

Exercise X.

Find the partial fractions of :—

1. $\frac{2x+3}{x^2+3x+2}$ **2.** $\frac{3x+1}{x^2-1}$

3. $\frac{7x-1}{6x^2-5x+1}$ **4.** $\frac{8x-33}{12x^2-7x-10}$

5. $\frac{x}{(3-x)(3+2x)}$ **6.** $\frac{1+x^2}{x(x^2-1)}$

7. $\frac{6x^3+5x^2-12x-17}{2x^2+x-6}$ **8.** $\frac{x^2}{x^3-2x^2-x+2}$

9. $\frac{24x^2-9x+16}{12x^2-11x-15}$ **10.** $\frac{px+q}{(x-a)(x+b)}$

11. $\frac{13-10x+x^2}{(x^2-5x+6)(x-1)}$ **12.** $\frac{11-10x+2x^2}{(x-4)(x-3)(x-2)(x-1)}$

13. $\frac{1}{(1-ax)(1-bx)(1-cx)}$

14. $\frac{2x^2}{(x^2+2)(x^2+3)}$ [*Hint.* Put $x^2 = y$; then split $\frac{2y}{(y+2)(y+3)}$.]

15. $\frac{x^9+3x^2-5x-22}{(x^3+2)(x^3+3)(x^3+4)}$

27. Case II. The denominator having linear factors one or more of which are repeated.

Ex. 1. Resolve $\frac{9x^3-24x^2+48x}{(x-2)^4(x+1)}$ into partial fractions.

Put $x-2=y$ so that $x=y+2$. Then

$$\frac{9x^3-24x^2+48x}{(x-2)^4(x+1)}=\frac{9(y+2)^3-24(y+2)^2+48(y+2)}{y^4(y+3)}$$

$$=\frac{72+60y+30y^2+9y^3}{y^4(3+y)}.$$

Let us divide the numerator by the factor $3+y$ of the denominator.

Thus
$$\begin{array}{r} 3+y)\ 72+60y+30y^2+9y^3\ (24+12y+6y^2+y^3 \\ \underline{72+24y} \\ 36y+3y^2 \\ \underline{36y+12y^2} \\ 18y^2+9y^3 \\ \underline{18y^2+6y^3} \\ 3y^3 \\ \underline{3y^3+y^4} \\ -y^4 \end{array}$$

$\therefore$ the given fraction

$$=\frac{1}{y^4}\left\{24+12y+6y^2+y^3-\frac{y^4}{3+y}\right\}$$

$$=\frac{24}{y^4}+\frac{12}{y^3}+\frac{6}{y^2}+\frac{1}{y}-\frac{1}{3+y}$$

$$=\frac{24}{(x-2)^4}+\frac{12}{(x-2)^3}+\frac{6}{(x-2)^2}+\frac{1}{x-2}-\frac{1}{x+1}.$$

Notice that we stop the process of division when the remainder is of the same degree as that of the repeated factor of the denominator.

Or thus. Noticing that all possible powers of the repeated factor can be used to get the required L. C. M., suppose that

$$\frac{9x^3-24x^2+48x}{(x-2)^4(x+1)}\equiv\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}+\frac{E}{(x-2)^4},$$

So that $9x^3-24x^2+48x \equiv A(x-2)^4+B(x+1)(x-2)^3$
$+C(x+1)(x-2)^2+D(x+1)(x-2)+E(x+1)$.

In this identity if we put $x=-1$, we get $A=-1$; and $x=2$ gives $E=24$.

Also comparing coefficients of x^4 we have $0=A+B$, whence $B=-A=1$;

Comparing coefficients of x^3 we have $9=-8A-5B+C$ whence $C=6$

and comparing coefficients of x^2 we have $-24=24+A+5B-3C+D$, whence $D=12$.

Thus the given fraction

$$=-\frac{1}{x+1}+\frac{1}{x-2}+\frac{6}{(x-2)^2}+\frac{12}{(x-2)^3}+\frac{24}{(x-2)^4}.$$

Ex. 2. Put into partial fractions $\dfrac{10x^3-42x^2+48x-35}{(x-2)^3(x+1)^2}$.

Let $\dfrac{10x^3-42x^2+48x-35}{(x-2)^3(x+1)^2}\equiv\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{(x-2)^3}+\dfrac{D}{(x+1)}+\dfrac{E}{(x+1)^2}$,

so that $10x^3-42x^2+48x-35\equiv(x+1)^2\{A(x-2)^2+B(x-2)+C\}+(x-2)^3\{D(x+1)+E\}$.

In this identity if we put $x=-1$, we get $E=5$; and $x=2$ gives $C=-3$. Also comparing the co-efficients of x^4, x^3 and absolute tems on both sides, we have

$$\left.\begin{aligned}0&=A+D\\10&=-2A+B-6D+E\\-35&=4A-2B+C-8D-8E\end{aligned}\right\}$$

i. e.,

$$\left.\begin{aligned}A+D&=0\\-2A+B-6D&=5\\4A-2B-8D&=8\end{aligned}\right\}$$

giving us $A=1$, $B=2$ and $D=-1$.

Hence the given fraction

$$\frac{1}{x-2}+\frac{2}{(x-2)^2}-\frac{3}{(x-2)^3}-\frac{1}{x+1}+\frac{5}{(x+1)^2}.$$

Or thus. Suppose that

$$\frac{10x^3-42x^2+48x-35}{(x-2)^3(x+1)^2}\equiv\frac{Ax^2+Bx+C}{(x-2)^3}+\frac{Dx+E}{(x+1)^2}$$

So that $10x^3-42x^2+48x-35\equiv(Ax^2+Bx+c)(x+1)^2+(Dx+E)(x-2.)^3$

Comparing the co-efficients of different powers of x on both sides we have

$$0 = A + D$$
$$10 = 2A + B - 6D + E$$
$$-42 = A + 2B + C + 12D - 6E$$
$$48 = B + 2C - 8D + 12E$$
$$-35 = C - 8E$$

From these equations $A = 1$, $B = -2$, $C = -3$, $D = -1$, $E = 4$.

Thus $$\frac{10x^3 - 42x^2 + 48x - 35}{(x-2)^3(x+1)^2} = \frac{x^2 - 2x - 3}{(x-2)^3} + \frac{-x+4}{(x+1)^2}.$$

To split $\frac{x^2 - 2x - 3}{(x-2)^3}$ put $x - 2 = y$ so that $x = y + 2$.

Then $$\frac{x^2 - 2x - 3}{(x-2)^3} = \frac{(y+2)^2 - 2(y+2) - 3}{y^3} = \frac{y^2 + 2y - 3}{y^3}$$
$$= \frac{1}{y} + \frac{2}{y^2} - \frac{3}{y^3}.$$
$$= \frac{1}{x-2} + \frac{2}{(x-2)^2} - \frac{3}{(x-2)^3}.$$

Also to split $\frac{-x+4}{(x+1)^2}$ put $x + 1 = z$ so that $x = z - 1$.

Then $$\frac{-x+4}{(x+1)^2} = \frac{-(z-1)+4}{z^2} = \frac{-z+5}{z^2}$$
$$= -\frac{1}{z} + \frac{5}{z^2}$$
$$= -\frac{1}{x+1} + \frac{5}{(x+1)^2}.$$

Hence the given fraction

$$= \frac{1}{x-2} + \frac{2}{(x-2)^2} - \frac{3}{(x-2)^3} - \frac{1}{x+1} + \frac{5}{(x+1)^2}.$$

Exercise XI.

Resolve into partial fractions.

1. $\dfrac{9}{(x+2)^2(x-1)}$.

2. $\dfrac{10-3x^2-3x^3+x^4}{(x-3)(x+1)^2}$.

3. $\dfrac{5x^3+6x^2+5x}{(x-1)(x+1)^4}$.

4. $\dfrac{10-8x^2+3x}{(x-1)^4}$.

5. $\dfrac{7x+4}{(2+3x)(1+x)^2}$.

6. $\dfrac{3-2x-x^2}{(1-4x)^2(1+x)}$.

7. $\dfrac{1-x+2x^2}{(1-x)^3}$.

8. $\dfrac{8x^4-10x^3-27x^2+16x+7}{(x+1)^2(x-1)^3}$.

9. $\dfrac{31-31x+10x^2-x^3}{(x-2)^2(x-3)^2}$.

10. $\dfrac{3x^4-x^3-2}{(x^2-2)^2(1-2x^2)}$.

28. Case III. The denominator having a quadratic or higher factor unrepeated.

Ex. 1. What are the partial fractions of $\dfrac{x^2}{x^3+8}$?

Now $\dfrac{x^2}{x^3+8} \equiv 1 - \dfrac{8}{x^3+8} \equiv 1 - \dfrac{8}{(x+2)(x^2-2x+4)}$.

Let $\dfrac{8}{(x+2)(x^2-2x+4)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2-2x+4}$,

so that $8 \equiv A(x^2-2x+4)+(Bx+C)(x+2)$.

In this identity put $x = -2$, then $A = \frac{2}{3}$; also equating the coefficients of x^2 and x on both sides, we have

$$0 = A + B$$

$$\text{and } 0 = -2A + 2B + C$$

whence we get $B = -\frac{2}{3}$ and $C = \frac{4}{3}$.

Hence the given fraction

$$= 1 - \frac{\frac{2}{3}}{x+2} - \frac{-\frac{2}{3}x+\frac{4}{3}}{x^2-2x+4}$$

$$= 1 - \frac{2}{3(x+2)} + \frac{2(x-4)}{3(x^2-2x+4)}.$$

Ex. 2. Find the partial fractions of
$$\frac{4x^3-12x^2+9x-6}{(x^2-3x+1)(x^3+2x^2-x+1)}.$$

Let $\dfrac{4x^3-12x^2+9x-6}{(x^2-3x+1)(x^3+2x^2-x+1)} \equiv \dfrac{Ax+B}{x^2-3x+1} + \dfrac{Cx^2+Dx+E}{x^3+2x^2-x+1}$

so that $4x^3-12x^2+9x-6 \equiv (Ax+B)(x^3+2x^2-x+1) + (Cx^2+Dx+E)(x^2-3x+1)$.

In this identity comparing the co-efficients of the like powers of x on both sides, we have

$$\begin{aligned} 0 &= A+C \\ 4 &= 2A+B-3C+D \\ -12 &= -A+2B+C-3D+E \\ 9 &= A-B+D-3E \\ -6 &= B+E. \end{aligned}$$

These equations give $A=2$, $B=-5$, $C=-2$, $D=-1$, $E=-1$.

Hence the given fraction
$$= \frac{2x-5}{x^2-3x+1} - \frac{2x^2-x-1}{x^3+2x^2-x+1}.$$

29. Case IV. The denominator having one or more quadratic or higher factors repeated.

Ex. Split $\dfrac{4x^4-21x^3+12x^2+43x+10}{(x^2-3x-2)^2(x-1)}$ into partial fractions.

Let $\dfrac{4x^4-21x^3+12x^2+43x+10}{(x^2-3x-2)^2(x-1)}$
$$\equiv \frac{Ax+B}{(x^2-3x-2)^2} + \frac{Cx+D}{x^2-3x+2} + \frac{E}{x-1},$$

so that $4x^4-21x^3+12x^2+43x+10$
$$\equiv \{Ax+B+(Cx+D)(x^2-3x-2)\}(x-1)+E(x^2-3x-2)^2.$$

Comparing the co-efficients of the like powers of x on both sides we get the equations

$$\begin{aligned} 4 &= C+E \\ -21 &= -4C+D-6E \\ 12 &= A+C-4D+5E \\ 43 &= A+B-2C-5D+12E \\ 10 &= -B+2D+4E, \end{aligned}$$

giving A=0, B=4, C=1, D=1, E=3.

Hence the given fraction

$$=\frac{4}{(x^2-3x-2)^2}+\frac{x+1}{x^2-3x-2}+\frac{3}{x-1}.$$

Exercise XII.

Put into partial fractions :—

1. $\frac{x^2}{x^3-1}$.

2. $\frac{7+3x-2x^2}{(x^2+1)(x+3)}$.

3. $\frac{1-x+x^2}{(x-1)^2(x^2+1)}$.

4. $\frac{4+3x+2x^2}{(1+x)(1+x+2x^2)}$

5. $\frac{2x+1}{(x-1)(x^2+1)}$.

6. $\frac{3-2x^2}{(3-3x+x^2)^2}$.

7. $\frac{x^2+3x+1}{x^4+x^2+1}$.

8. $\frac{2+x-2x^3+x^4}{(1+x)(1+x^2)^2}$.

9. $\frac{1}{(x^2+x+2)(x^2+x+1)}$.

10. $\frac{2-3x}{(x-1)^2(x^2-x+1)}$.

11. $\frac{x^5}{x^4+5x^2+4}$

12. $\frac{1}{x(x^2+1)^3}$.

CHAPTER V

Arithmetical Progressions.

30. Definitions. Consider the following different collection of numbers:

(1) 1, 2, 3, 4, 5,.....
(2) 4, 7, 10, 13, 16.....
(3) 1, 3, 9, 27, 81.....
(4) 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$,.....
(5) 2, 7, 10, 15, 34,.....

Each of the first four collections are framed according a particular *definite* law. Each term of (1) is got by adding 1 to the preceding term In (2) any term increased by 3 gives the next term. Any term in (3) is three-times the preceding term. The reciprocals of the terms in (1) constitute the corresponding terms in (4). But in (5) no such law can be discovered.

Any collection (or *set*, as it is called) of numbers following a certain *definite* law is called a **sequence**.

Series is a set of numbers following one another according to a definite law and connected together by plus or minus signs. Thus

(a) $2+6+10+14+$.....
(b) $4+16+64+$.....
(c) $3-5+7-9+11$.....

are examples of series.

If the number of terms in a series is finite *i. e.*, if a series ends in a certain term, the series is called *terminating* or *finite.*

But if a series is such that each term is followed by another and there is no last term it is called *infinite.*

Arithmetical Progression is a *series* or sequence in which terms increase or decrease by a common difference. For example the sequences

2, 7, 12, 17, 22,.....
16, 12, 8, 4, 0, -4, -8,.....
a, $a+d$, $a+2d$, $a+3d$,.....

are all in Arithmetical Progression.

We shall often write A P. for Arithmetical Progression ; and the letter T will be used for the word term, so that T_1, T_2, T_3,.....T_p will mean the first term, second term, third term..... *pth* term respectively.

31. To find the nth term of an A. P. whose first term and common difference are given. Suppose that the first term is a and the common difference is d. Then

$T_1 = a$; $T_2 = a + d = a + (2-1)d$;

$T_3 = a + 2d = a + (3-1)d$; $T_4 = a + 3d = a + (4-1)d$;

and so on

It is evident that the co-efficient of d in any term is the suffix of the corresponding T decreased by unity. Hence

$$T_n = a + (n-1)d.$$

If l be the value of the nth term of an A. P. whose first term is a and the common difference d, then we have

$$l = a + (n-1)d.$$

Cor. The nth term of the sequence a, $a-d$, $a-2d$, $a-3d$ etc. is $a-(n-1)d$.

Ex. 1. Find the 21st term of the A. P. 5, 8, 11,.....

Here $a = 5$ and $d = 3$

$\therefore \quad T_{21} = a + 20d = 5 + 20 \times 3 = 65.$

Ex. 2. Find the 9th term of an A. P. whose 4th term is 15 and 13th term is 33.

Let a be the first and d the common difference of the A. P.

Then we have $15 = T_4 = a + 3d$

and $33 = T_{13} = a + 12d.$

These two equations simultaneously give $a = 9$ and $d = 2$.

$\therefore \quad T_9 = a + 8d = 9 + 8 \times 2 = 25$

Ex. 3. What term of the A. P. 3, 10, 17, 24,.....is 136.

Let 136 be the rth term. Then since $a = 3$ and $d = 7$, we have

$$136 = T_r = a + (r-1)d$$

$$=3+(r-1)7$$

whence we get $r=20$. Hence 136 is the 20th term.

Ex. 4. Find the A. P. whose nth term is $2n-7$.

In the given expression if we put $n=1, 2, 3, 4,\ldots\ldots n$ in succession we get the first, second, third, fourth terms of the A. P.

Hence $T_1=2\cdot1-7=-5$
$T_2=2\cdot2-7=-3$
$T_3=2\cdot3-7=-1$
$T_4=2.4-7=\quad 1.$

Thus the series is $-5, -3, -1, 1, 3,\ldots\ldots, 2n-7.$

32. To find the sum of n terms of an A. P.

Suppose that S is the sum of n terms of the A. P. whose first term is a and the common difference is d. Then

$S=a+(a+d)+(a+2d)+\ldots\ldots+(a+n-1d).$

If l be the value of the nth term. then

$$a+n-1d=l$$
$$a+n-2d=l-d$$
$$a+n-3d=l-2d, \text{ and so on.}$$

Thus the above equation is the same as

$$S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l\ldots(1)$$

Writing the right-hand side of (1) in the reverse order, we get

$$S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a\ldots\ldots(2)$$

Adding (1) and (2) together, we get

$$2S=(a+l)+(a+l)+(a+l)\ldots\ldots\text{to } n \text{ terms}$$
$$=n(a+l)$$

$$\therefore\ S=\frac{n}{2}(a+l)\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

Also substituting the value of l, we get

$$S=\frac{n}{2}(2a+n-1d)\ldots\ldots\ldots\ldots(B)$$

The formula (A) gives the sum of n terms when the first and the last terms are known ; whereas the formula (B) gives the sum of n terms when the first term and the common difference are known.

Ex. 1. Find the sum of the first 23 terms of the A. P. 3, 6, 9, 12,……

Here $a=d=3$ and $n=23$. Thus if S be the required sum, then

$$S=\frac{n}{2}\{2a+(n-1)d\}$$
$$=\frac{23}{2}(2\cdot3+22\cdot3)$$
$$=828.$$

Ex. 2. Find the sum of 52 terms of the A. P. whose 7th term is 24 and 12th term is 19.

Let a be the first term and d the common difference. Then

$$24=T_7=a+6d$$
$$19=T_{12}=a+11d$$

whence we get $a=30$, and $d=-1$

$$\therefore \quad l=T_{52}=30+51\cdot(-1)$$
$$=-21.$$

Hence if S be the required sum, we get

$$S=\frac{n}{2}(a+l)$$
$$=\frac{52}{2}(30-21)$$
$$=234.$$

Ex 3. Find the number of terms and the common difference of the A. P. whose sum is 210 and whose first and last terms are 7 and 35 respectively.

Here we have $S=210$. $a=7$, $l=35$, to find n and d.

Thus the formula $S=\frac{n}{2}(a+l)$ gives

$$210=\frac{n}{2}(7+35)$$

$$\therefore \quad n=10\ ;$$

and the formula $l=a+(n-1)d$ gives

$$35=7+9d$$

$$\therefore \quad d=\frac{28}{9}=3\frac{1}{9}.$$

Hence the number of terms is 10 and the common difference is $3\frac{1}{9}$

Ex 4 If the sum of n terms of a series be n^2+2n, show that it is an A P. and find it.

The sum of n terms $= n^2 + 2n$

$\therefore$ The sum of $n-1$ terms $= (n-1)^2 + 2(n-1)$

$= n^2 - 1$

$\therefore$ the nth term $= (n^2 + 2n) - (n^2 - 1)$

$= 2n + 1.$

Putting $n = 1, 2, 3, 4, \ldots$ in succesion we get the successive terms of the series. Thus the series is

$3 + 5 + 7 + 9 + 11 + \ldots\ldots$

which is an A. P. with common difference 2.

Exercise XIII

1. The first term of an A. P. is 7 and the common difference is 2, find the 7th, 22nd and $(n-1)$th terms.

2. Find the 29th, 47th and $(r+3)$th terms of the A. P. 39, 35, 31, 27,.....

3. What terms of the A. P. 3, 11, 19, 27. etc are 67, 99, $8n-5$, and $16r-21$?

4. The first term of an A. P. is 3 and 7th term 39, find its 5th, 12th and pth terms

5. The sum of the 2nd and 7th terms of an A. P. is 25, and the sum of the 5th and 18th terms is 67. Find the A. P.

6. The 3rd and the 9th terms of an A. P. are $-2\frac{1}{4}$ and $-5\frac{3}{7}$, find its 15th term.

7. The 3rd term of an A. P. is 3 and the 11th term is -21 ; what term is zero ?

8. The $(m+2)$th term of an A. P. is $5m+1$; find its 4th, 17th and mth terms.

9. Sum the following series :—

(*i*) $2+4+6+8+\ldots\ldots$to 17 terms.

(*ii*) $3+8+13+18+\ldots\ldots$to 30 terms.

(*iii*) $1+\frac{3}{3}+2+\frac{5}{3}+3+\ldots\ldots$to 27 terms.

(*iv*) $7\frac{1}{4}+9\frac{1}{4}\frac{5}{4}+11\frac{1}{4}\frac{9}{4}+13\frac{2}{4}\frac{3}{4}+\ldots\ldots$ to 12 terms.

(*v*) $-81-78-75-72-\ldots\ldots$to 55 terms.

(*vi*) $21^{\circ}2+18\,3+15^{\circ}4+12^{\circ}5+\ldots\ldots$to 47 terms.

(*vii*) $\frac{1}{x}+\frac{2}{x}+\frac{3}{x}+\frac{4}{x}+\ldots\ldots$to 20 terms.

(*viii*) $\frac{4}{3}+\frac{5}{4}+\frac{7}{8}+\frac{13}{12}+\ldots\ldots$to n terms.

(*ix*) $(n-1)+(n-2)+(n-3)+\ldots$to n terms.

(P. U. 1925)

(v) $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+$.....to n terms.

10 Find the sum of the series :—

(i) $17\frac{1}{3}+14\frac{1}{3}+10\frac{2}{3}+\ldots-59\frac{1}{3}$

(ii) $\frac{1}{1+\sqrt{3}}-\frac{1}{2}+\frac{1}{1-\sqrt{3}}+\ldots\ldots-\frac{1+16\sqrt{3}}{2}$.

(iii) $\frac{1}{n}+\frac{n+1}{n}+\frac{2n+1}{n}+\ldots\ldots+\frac{n^2-n+1}{n}$.

(P. U. 1917)

(iv) $105+103+101+\ldots\ldots+5$.

(v) $\frac{1}{7}+\frac{3}{7}+\frac{5}{7}+\ldots\ldots+\frac{109}{7}$.

11. Find the sum of (*i*) the first 60 natural numbere, (*ii*) the first 45 odd numbers (*iii*) first n even numbers.

12. The 5th and 9th terms of an A. P. are 9 and 12 respectively. Find the sum of the first 17 terms.

13. The first, third and the nth terms of an A. P. are a, b, c respectively. Find the sum of the first n terms.

14. How many terms of the A. P.

(*i*) $4+4\frac{1}{5}+4\frac{1}{4}+\ldots\ldots$give the sum 79 ?

(*ii*) $6+9+12+\ldots\ldots$give the sum 897 ?

(*iii*) $27+25+23+21+\ldots\ldots$give the sum 171 ?

Explain the double answers.

15. The first term of an A. P. is 3 and the common difference 2. If the sum of the first n terms be 1935, find n.

16. How many terms of $17+14+11+\ldots\ldots$amount to -190 ?

17. Find the sum of all the numbers of 2 digits each, which are divisible by 6.

18. Find the sum of the n terms of the A. P. whose $(r+9)$th term is $4r+5$.

19. The sum of n terms of an A. P. is $2n^2+7n$ Find the first term and the common difference.

20. The sum of n terms of an A. P. is an^2+bn+c. Find the first term and the common difference.

21. In an A. P. the ratio of the sum of the first 5 terms to the sum of the next 5 terms is 9 : 29 and the ratio of the 3rd term to the 7th term is 9 : 25. Find the sum of the n terms.

22. Sum up the series :-

(*i*) $\frac{1}{1+\sqrt{x}}+\frac{1}{1-x}+\frac{1}{1-\sqrt{x}}+$.....to n terms.

(*ii*) $(a+x)^2+(a^2+x^2)+(a-x)^2+$.....to 15 terms.

(*iii*) $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+$.....to n terms.

(*iv*) $\frac{3n-2}{n}+\frac{3n-5}{n}+\frac{3n-8}{n}+$.....to n terms.

(*v*) $\frac{2x-y}{x+y}+\frac{3x-+y}{x+y}+\frac{+x-7y}{x+y}+$ ...to $(x+y)$ terms.

23. A polygon has its interior angles in A. P. If the least angle be of 120° and the common difference 5°, find the number of sides.

24. A debt can be discharged in three years by paying Rs. 5 the first month, Rs. 8 the second, Rs. 11 the third and so on. Find the last payment and the amount of debt.

25. A class consists of boys whose ages form an A. P. with common difference four months. If the smallest of them is just eight years old, and the total sum of ages is 168 years, find the number of boys.

33. Arithmetic Mean. Of three quantites in A. P. the middle one is called the *Arithmetic Mean* between the other two.

Of a number of quantities in A. P. all the intermediate ones are called the *Arithmetic Means* between the first and the last.

Arithmetic Mean for shortness is written as A. M.

I. *To find the Arithmetic Mean between two given quantities.* If a and b be the given quantities, suppose that x is the required A. M. Then since a, x, b are in A. P., we have

$$x-a=b-x$$

or $$2x=a+b$$

or $$x=\frac{a+b}{2}.$$

Thus the A. M. between two quantities is half the sum of those quantities.

II. *To insert n Arithmetic Means between two given quantities.* If a and b be the given quantities,

suppose that $x_1, x_2\ x_3, \ldots\ldots x_n$ are the required arithmetic means. Then

$$a, x_1, x_2, x_3, \ldots\ldots, x_n, b$$

form an A. P. of $n+2$ terms. Let d be the common difference of this A. P.

Thus $b = a + (n+1)d$.

$$\therefore \quad d = \frac{b-a}{n+1}$$

Hence $x_1 = a + d = a + \frac{b-a}{n+1}$,

$$x_2 = a + 2d = a + 2\frac{b-a}{n+1},$$

$$x_3 = a + 3d = a + 3\frac{b-a}{n+1},$$

. . . . . .

$$x_n = a + nd = a + n\frac{b-a}{n+1}.$$

Ex. 1. Insert 7 Arithmetic Means between 4 and 22.

From above d the common difference

$$= \frac{b-a}{n+1} = \frac{22-4}{7+1} = \frac{9}{4} = 2\tfrac{1}{4}.$$

Hence the required means are $6\frac{1}{4}$, $8\frac{1}{2}$, $10\frac{3}{4}$, 13, $15\frac{1}{4}$, $17\frac{1}{2}$, $19\frac{3}{4}$.

Ex. 2. Between 1 and 31 are inserted m Arithmetic Means such that the 7th mean : $(m-1)$th mean $= 5 : 9$. Find m.

Here $d = \frac{31-1}{m+1} = \frac{30}{m+1}$.

Hence 7th mean $= 1 + \frac{7.30}{m+1} = 1 + \frac{210}{m+1} = \frac{m+211}{m+1}$

and $(m-1)$th mean $= 1 + \frac{30(m-1)}{m+1} = \frac{31m-29}{m+1}$

$$\therefore \quad \frac{m+211}{m+1} : \frac{31m-29}{m+1} = 5 : 9$$

or $9(m+211) = 5(31m-29)$

whence $m = 14$.

Exercise XIV

1. Find 6 arithmetic means between 4 and 25.

2. Insert 12 arithmetic means between 21 and -5.

3. Insert m arithmetic means between m^2 and 1.

4. Insert r arithmetic means between $5a-7b$ and $7a-5b$.

5. Show that the sum of n arithmetic means inserted between two quantities equals to n times the single mean between them.

6. Between 3 and 39 are inserted p means such that the ratio between the 6th mean and the 11th mean is 33 : 53. Find the value of p.

7. An A. P. of n terms has a the first term and b the last term. Between every two successive terms is inserted an A. M. Show that all these means form an A. P. Also show that the sum of the original A. P. bears to the sum of the means the ratio $n : (n-1)$.

8. Show that the ratio between the sum of n arithmetic means and the sum of m arithmetic means between two quantities is $n : m$.

9. Find n if the A. M. between a and b be

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}.$$

34. We shall conclude this chapter by solving some miscellaneous examples of useful application.

Ex. 1. The sum of 3 terms in A. P. is 45 and their product is 3135. Find them.

Suppose that the terms are $a-d$, a, $a+d$. Then

$$45=(a-d)+a+(a+d)$$
$$=3a$$
$$\therefore \quad a=15;$$

and $$3135=a(a^2-d^2)=15(225-d^2)$$

$$\therefore \quad d=\pm 4.$$

Hence the terms are 11, 15, 19 or 19, 15, 11.

Ex. 2. Four numbers in A. P. are such that their sum is 74 and the sum of their squares is 1374. What are the numbers ?

Let the numbers be $a-3d$, $a-d$, $a+d$, $a+3d$. Then

$$74=(a-3d)+(a-d)+(a+d)+(a+3d)$$
$$=4a$$
$$\therefore \quad a=\tfrac{37}{2};$$

and $$1374=(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2$$
$$=4(a^2+5d^2)$$

$$= 4\left(\frac{1369}{4} + 5d^2\right)$$

$\therefore \quad d = \pm\frac{1}{2}$.

Hence the numbers are 17, 18, 19, 20.

Note. From the last two examples notice the selection of algebraic numbers in A. P. in two different cases.

Verification : Sum of numbers $= 17+18+19+20 = 74$,

Sum of squares $= 17^2+18^2+19^2+20^2 = 1374$.

Ex. 3. In the series $1+(3+5)+(7+9+11)+\ldots\ldots$

Find the sum of the terms in the nth group.

The number of terms in the first group is 1 ;

" " " " " " second " " 2 ;

" " " " " " third " " 3 ;

and so on.

∴ The number of terms in the nth group is n.

Also the last term in the first group is the 1st term of the series 1, 3, 5, 7, 9,.....

Also the last term in the second group is the $(1+2)$th term of the same series ;

Also the last term in the third group is the $(1+2+3)$th term of the same series and so on.

∴ The last term in the nth group is the $(1+2+3+\ldots\ldots+n)$th term of the same series ;

i.e., the $\frac{n(n+1)}{2}$th term of the same series.

Now this term by the help of the formula $l = a+(n-1)d$ comes out to be n^2+n-1. Also since the group contains n terms, from the same formula we get the first term of the group n^2-n+1.

$$\text{Thus the sum} = \frac{n}{2}\ (\text{the first term} + \text{the last term})$$

$$= \frac{n}{2}[(n^2-n+1)+(n^2+n-1)]$$

$$= n^3.$$

Ex. 4. In an A. P. the sum of x terms is y and the sum of y terms is x. Find the sum of $x+y$ terms.

Let a be the first term and d the common difference.

Then the required sum $= \frac{x+y}{2}\{2a+(x+y-1)d\}$.

Now $y=\frac{x}{2}\{2a+(x-1)d\}=ax+\frac{d}{2}x(x-1)$

and $x=\frac{y}{2}\{2a+(y-1)d\}=ay+\frac{d}{2}y(y-1)$

$\therefore$ by subtraction $y-x=a(x-y)+\frac{d}{2}(x-y)(x+y-1)$

or $-1=a+\frac{d}{2}(x+y-1)$

or $-2=2a+d(x+y-1)$.

Hence the sum $\frac{x+y}{2}\{2a+(x+y-1)d\}$

$=\frac{x+y}{2}(-2)=-(x+y)$.

Ex. 5. In an A. P. the sum of the first m terms is to the sum of its first n terms as $m^2:n^2$; show that the common difference is twice the first term.

If a be the first term and d the common difference, then by the given condition we have

$$\frac{\frac{1}{2}m\{2a+(m-1)d\}}{\frac{1}{2}n\{2a+(n-1)d\}}=\frac{m^2}{n^2}.$$

or $\frac{2a+(m-1)d}{2a+(n-1)d}=\frac{m}{n}$

or $2amn+mn(m-1)d=2amn+mn(n-1)d$

or $(m-n)d=2a(m-n)$

$\therefore\ d=2a$.

Ex. 6. The sums of n terms of two Arithmetic Progressions are in the ratio of $3n+8:7n+15$: find the ratio between their 12th terms

Let the two series respectively have their first terms a and a', and the common differences d and d'. Then

$$\frac{\frac{1}{2}n\{2a+(n-1)d\}}{\frac{1}{2}n\{2a'+(n-1)d'\}}=\frac{3n+8}{7n+15}$$

or $\frac{2a+(n-1)d}{2a'+(n-1)d'}=\frac{3n+8}{7n+15}$ .....(1)

To find the ratio between the 12th terms we have to find the value of

$\frac{a+11d}{a'+11d'}$ *i. e.* of $\frac{2a+(23-1)d}{2a'+(23-1)d'}$.

Hence putting $n=23$ in (1) we get the required ratio $\frac{7}{8}$ or $\frac{7}{16}$.

Ex. 7. If $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ be in A. P. show that $\frac{1}{b-c}$, $\frac{1}{c-a}$, $\frac{1}{a-b}$ are also in A. P.

clearly $\frac{1}{b-c}$, $\frac{1}{c-a}$, $\frac{1}{a-b}$ will be in A. P. if

$$\frac{1}{b-c}+\frac{1}{a-b}=\frac{2}{c-a}$$

i. e., if $(c-a)(a-c)=2(b-c)(a-b)$

i. e., if $2ac-a^2-c^2=2ab+2bc-2ac-2b^2$

i. e., if $2c^2+2a^2-4ac=a^2+b^2-2ab+b^2+c^2-2bc$

i. e., if $2(c-a)^2=(a-b)^2+(b-c)^2$

which is true.

Ex. 8 If $S_1, S_2, S_3 \ldots\ldots S_m$ be the sums of n terms of m Arithmetic Progressions whose first terms are 1, 2, 3,..... m and common differences are 1, 3, 5,..... $2m-1$ respectively, show that

$$S_1+S_2+S_3+\ldots\ldots+S_m=\tfrac{1}{2}mn(mn+1).$$

We have $S_1=\frac{n}{2}\{2+(n-1).1\}=\frac{1}{2}n(n+1)$

$S_2=\frac{n}{2}\{4+(n-1).3\}=\frac{1}{2}n(3n+1)$

$S_3=\frac{n}{2}\{6+(n-1).5\}=\frac{1}{2}n(5n+1)$

....

$S_m=\frac{n}{2}\{2m+(n-1)(2m-1)\}$

$=\frac{1}{2}n\{(2m-1)n+1\}$

$\therefore\ S_1+S_2+S_3+\ldots\ldots+S_m$

$=\frac{1}{2}n\{(n+1)+(3n+1)+(5n+1)+\ldots\ldots+(\overline{2m-1})n+1\}$

$=\frac{1}{2}n\{(n+3n+5n+\ldots\ldots+2m-1)n+1+1+1+\ldots\ldots$ to m terms$\}$

$=\frac{1}{2}n\{n(1+3+5+\ldots\ldots 2m-1)+m\}$

$=\frac{1}{2}n\{n\frac{m}{2}(1+2m-1)+m\}$

$=\frac{1}{2}n\{m^2n+m\}$

$=\frac{1}{2}mn(mn+1).$

Exercise XV.

1. The sum of three numbers in A. P. is 9 and the sum of their squares is 35. Find the numbers.

2. Find three numbers in A. P. whose sum is 33 and whose product is 1155.

3. What four numbers in A. P. have their sum 40 and the sum of their squares 420 ?

4 What four numbers in A. P. have their sum 10 and their product 24 ?

5. Divide 28 into 4 parts in A. P. such that the product of the 1st and 3rd is to the product of the 2nd and 4th as 8 : 15.

6. In the equation $(x-l+m)^2+(x-m+n)^2=0$ show that the roots will be equal if l, m, n are in A. P.

7. In an A. P. if the xth term is y and the yth term is x then the rth term is $x+y-r$

8. If the sum of p terms of an A. P. is the same as of q terms, show that the sum of $p+q$ terms is zero.

9. If the pth term of on A. P be a and the qth term b, show that the sum of $p+q$ term is

$$\frac{p+q}{2}\left\{a+b+\frac{a-b}{p-q}\right\}$$

10. If the pth, qth, rth terms of an A. P. be a, b, c, respectively, show that

$$p(b-c)+q(c-a)+r(a-b)=0$$

11. If the sum of 25 terms of an A. P. be 235, find the 13th term.

12. If the 17th term of an A. P. be 13 find the sum of 33 terms

13. The sums of n terms of two series in A. P. are in the ratio of $2n-7 : 4n+1$; find the ratio of their 9th terms.

14. The sums of n terms of two Arithmetic Progressions are as $13-7n : 1+3n$; find the ratio of their 3rd terms

15. The sums of n terms of two Arithmetic series are as $7n+1 : 3n+29$; find the ratio of their 7th terms and of 11th terms.

16. In an A. P. the sum of m terms is to the sum of n terms as $m^2 : n^2$; show that the ratio of the mth term to

the nth term is $2n-1$: $2n-1$.

17. In the series $1+(2+3)+(4+5+6)+(7+8+9+10)+$.....find the sum of the terms in the nth group.

18. In the series

$1+(2+3++)+(5+6+7+8+9)+(10+11+12+13+14+15+16)+$.. ...find the sum of the nth group of terms.

19. In an A. P. the first term is p, the 5th term q and the last term r ; show that the sum of the A. P. is

$$\frac{p+r}{2} \cdot \frac{5p-q-4r}{p-q}$$

20. In a series the sum of n terms is $3n^2+4n$. Show that it is an A. P. Find its first term and common difference.

21. If an A. P consists of an odd number of terms, show that its sum equals the middle term multiplied by the number of terms.

22. If a, b, c, are in A. P. Show that

$\frac{1}{bc}$, $\frac{1}{ca}$, $\frac{1}{ab}$ are also in A. P.

23. If a, b, c are in A. P. Show that

$\frac{a}{bc}(b+c)$, $\frac{b}{ca}(c+a)$, $\frac{c}{ab}(a+b)$ are also in A. P.

24 If a, b, c are in A. P. Show that

$a^2(b+c)$, $b^2(c+a)$, $c^2(a+b)$ are also in A. P.

25 If a, b, c are in A. P. Show that

$(b+c)^2-a^2$, $(c+a)^2-b^2$, $(a+b)^2-c^2$ are also in A. P.

26. If a, b, c are in A. P. Show that

$\frac{1}{\sqrt{b}+\sqrt{c}}$, $\frac{1}{\sqrt{c}+\sqrt{a}}$, $\frac{1}{\sqrt{a}+\sqrt{b}}$ are also in A. P.

27. If $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in A. P. Show that

$\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ are also in A. P.

28. If $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A. P. Show that

$a(b+c)$, $b(c+a)$, $c(a+b)$ are also in A. P.

29. Show that the equation

$(b-c)x^2+(c-a)x+(a-b)=0$ will have equal roots if a, b, c are in A. P.

30. The sums of n, $2n$, $3n$ terms of an A. P. are S_1, S_2, S_3 ; show that $S_3=3(S_2-S_1)$.

31. The sums of n terms of three Arithmetical series are S_1, S_2, S_3. The first terms of the series are unity and the common differences are 1, 3, 5. Show that S_1, S_2, S_3 are in A. P.

32. There are n Arithmetical progressions each begining with unity, whose common differences are 1, 2, 3,....., n. Show that the sum of their nth terms is $\frac{1}{2}n(n^2+1)$.

33. If S_1, S_2, S_3,.....S_m be the sums of n terms of m Arithmetic series whose first terms as well as the common differences are 1, 2, 3,..... , m, show that

$$S_1+S_2+S_3+.....+S_m=\tfrac{1}{4}mn(m+1)(n+1).$$

34. Sum the series $2-3++-5+6$.....to +0 terms.

[The given series $=(2++6+$.....to 20 terms) $-(3+5+7+$.....to 20 terms)].

35. Sum the series $1+2-3+4+5-6+7+8-9+$... to $3n$ terms.

CHAPTER VI

Geometrical Progressions

35. Definitions The type of *series* or sequence in which the successive terms increase or decrease by a constant factor is called the **Geometrical Progression** : and the constant factor is called the **common ratio.**

Thus (1) 2, 4, 8, 16, 32,.....

(2) 1, $-\frac{2}{3}$, $\frac{4}{9}$, $-\frac{8}{27}$, $\frac{16}{81}$.....

(3) a, ar, ar^2, ar^3,.. .....

are all illustrations of the Geometrical Progression. In (1) the common ratio is 2 ; in (2) it is $-\frac{2}{3}$: and in (3) it is r.

For the sake of brevity G. P. will be often written for Geometrical Progression.

Evidently any term of a G. P. can be obtained by multiplying the preceding term or by dividing the succeeding term by the common ratio.

36. To find the nth term of a G. P. whose first term and the common ratio are known. Suppose that a is the first term and r the common ratio. Then

$T_1 = a$ $\qquad$ $T_2 = ar$

$T_3 = ar^2$ $\qquad$ $T_4 = ar^3$

and so on.

Clearly in any term the index of r is the same as the suffix of T decreased by unity. Hence $T_n = ar^{n-1}$.

Ex. 1. What is the 5th term of the G. P. whose first term is $6\frac{3}{4}$ and the common ratio is $\frac{2}{3}$.

Since $T_n = ar^{n-1}$, we get

$$T_5 = 6\tfrac{3}{4} \times (\tfrac{2}{3})^4 = \tfrac{27}{4} \times \tfrac{16}{81} = \tfrac{4}{3}.$$

Ex. 2. In a G. P. the 4th term is $6\frac{1}{4}$ and the 7th term is $97\frac{2}{3}\frac{1}{2}$; find the 3rd term.

We have $\frac{25}{4} = T_4 = ar^3$

and $3\frac{1}{3}\frac{2}{5} = T_7 = ar^6$

Dividing, $r^3 = 3\frac{1}{3}\frac{2}{5} \times \frac{4}{25} = \frac{125}{64} = (\frac{5}{4})^3$

$\therefore$ $r = \frac{5}{4}$.

and hence $$a = \frac{25}{4} \times \frac{1}{r^2} = \frac{25}{4} \times \frac{8}{125} = \frac{2}{5}$$

Thus $T_3 = ar^2 = \frac{2}{5} \times \frac{25}{16} = \frac{5}{8}$.

37. Sum of a given number of terms in G. P.

Let us find the sum of n terms of the G. P. whose first term is a and the common ratio r.

If S denotes the required sum, we have

$$S = a + ar + ar^2 + \ldots\ldots\ldots + ar^{n-1}$$

$$\therefore \quad Sr = ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^n$$

By subtraction

$$Sr - S = ar^n - a$$

or $$S(r-1) = a(r^n - 1)$$

$$\therefore \quad S = a\,\frac{r^n - 1}{r - 1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (I)$$

which may also be written as

$$S = a\,\frac{1 - r^n}{1 - r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (II)$$

Cor. If l be the last term then $l = ar^{n-1}$. Thus (I) becomes $S = \dfrac{lr - a}{r - 1}$ and (II) becomes $S = \dfrac{a - lr}{1 - r}$.

Note. The formula (I) is used when $r > 1$ and (II) when $r < 1$.

38. Sum of the Infinite Geometrical Series.

The number of terms in the series

$$a + ar + ar^2 + ar^3 + \ldots\ldots\ldots + ar^n$$

depends upon the value of n. As we go on increasing the value of n the number of terms will become greater and greater. The series will become infinite when n is very great. (Art. 19) Then, since there will be no last term, it will be written as

$$a + ar + ar^2 + ar^3 + \ldots\ldots\ldots \text{ ...to } \infty.$$

To find the sum of an infinite G. P. we shall take r numerically less than one. Denoting the sum of n terms by S_n, we know that

$$S_n = a\,\frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

or $$\frac{a}{1 - r} - S_n = \frac{ar^n}{1 - r}.$$

Now, since r is less than unity, r^n decreases in value as n increases. It means that as n becomes greater and greater, $\dfrac{ar^n}{1 - r}$ becomes smaller and smaller. Thus by taking n sufficiently large, the difference between $\dfrac{a}{1 - r}$ and S_n

can be made as small as we like, *i. e.*, very near to zero. Then S_n is very nearly equal to $\frac{a}{1-r}$ which is called *the limiting sum* of the infinite series. The limiting sum of an infinite series is, for the sake of brevity, called its sum.

Thus the sum of $a+ar+ar^2+ar^3+\ldots\ldots\ldots\ldots$to ∞ is $\frac{a}{1-r}$ when r is less than one.

Note 1 If the sum of an infinite series is finite, the series is said to be **convergent**. Thus we see that the infinite series in G. P. is convergent when the common ratio is numerically less than one.

Note 2 If, however, r is greater than one or equal to one, the term $\frac{a}{1-r}$ increases as n increases. Therefore, the sum of the series increases within limits and there is no definite sum. Such series are called divergent series.

Note 3. The limiting sum should be carefully distinguished from the actual sum. There is no actual sum of an infinite series since there is no actual number infinity.

Ex. 1. Find the sum of 12 terms of the G. P. whose 3rd term is 3 and 5th term is 9.

If a be the first term and r the common ratio, then

$$3 = T_3 = ar^2$$

and $$9 = T_5 = ar^4$$

whence we get $r = \pm\sqrt{3}$ and $a = 1$.

Thus the series is $1+\sqrt{3}+3+3\sqrt{3}+\ldots\ldots$

or $1-\sqrt{3}+3-3\sqrt{3}+\ldots\ldots$

In the first case the sum $= \frac{(\sqrt{3})^{12}-1}{\sqrt{3}-1} = \frac{728}{\sqrt{3}-1}$

$$= \frac{728(\sqrt{3}+1)}{3-1} = 364(\sqrt{3}+1);$$

and in the second case the sum $= \frac{(-\sqrt{3})^{12}-1}{-\sqrt{3}-1} = -\frac{728}{\sqrt{3}+1}$

$$= -\frac{728(\sqrt{3}-1)}{3-1} = 364(1-\sqrt{3}).$$

Ex. 2. Sum the series

$$\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\ldots\ldots\ldots\text{to } \infty.$$

Since the common ratio is $\frac{1}{3}$ which is less than unity we see that the formula $S = \frac{a}{1-r}$ will apply.

Hence the required sum is $\frac{\sqrt{3}}{1-\frac{1}{3}} = \frac{3\sqrt{3}}{2}$.

Ex. 3. Find the value of $\cdot\dot{3}\dot{4}\dot{5}$.

Now $\cdot\dot{3}\dot{4}\dot{5} = \cdot 345454545\ldots\ldots\ldots\ldots$ to infinity

$= \cdot 3 + \cdot 045 + \cdot 00045 + \cdot 0000045 + \ldots$ to infinity

$= \frac{3}{10} + \frac{45}{1000} + \frac{45}{100000} + \frac{45}{10000000} + \ldots$ to infinity

which excluding the first term is, a G. P.

$$\therefore \quad \cdot\dot{3}\dot{4}\dot{5} = \frac{3}{10} + \frac{\frac{45}{1000}}{1-\frac{1}{100}} = \frac{3}{10} + \frac{45}{990} = \frac{345}{990} = \frac{23}{66}.$$

Notice the same result as can be obtained arithmetically. In fact the arithmetical formula is based upon this very principle.

Exercise XVI

1. In a G. P. the first term is 32 and the common ratio is $\frac{1}{2}$, find the 5th term.

2. Find the 9th term of the series $7 + \frac{7}{3} + \frac{7}{9} + \ldots\ldots\ldots$

3. In the series $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + \ldots$find the 6th and 11th terms.

4. Find the 10th and nth terms of a G. P. whose 2nd term is -1 and 5th term is $\frac{1}{8}$.

5. The $(n+3)$th term of a G. P. is 2^n, find its 3rd and 7th terms.

6. What term of the series $9+3+1+\ldots\ldots$is $\frac{1}{729}$?

7. The 3rd and 9th terms of a G. P. are reciprocal of one another ; what term is unity ?

8. In the G. P. $1+3+9\ldots$find the 7th term and the sum of 7 terms.

9. Find the sum of 12 terms of the series $1 - \frac{2}{3} + \frac{4}{9} - \ldots$, find also the 12th term.

10. In a G.P. the first term is $\frac{3}{5}$ and the 4th term $\frac{1}{135}$, find the sum of its first 10 terms.

11. Sum the following series :—

(*i*) $1+2+4+\ldots\ldots$to 6 terms.

(*ii*) $9+6+4+\ldots\ldots$to 10 terms.

(*iii*) $\frac{5}{2} - 1 + \frac{2}{5} - \ldots\ldots$to 8 terms.

(*iv*) $1 + \sqrt{2} + 2 + \ldots\ldots$to 9 terms.

(v) $\frac{1}{3} - \frac{1}{\sqrt{3}} + 1 - \ldots$ ...to 18 terms.

(vi) $\frac{1}{\sqrt{2}} + \frac{1}{1+\sqrt{2}} + \frac{2}{4+3\sqrt{2}} + \ldots\ldots$to n terms.

(vii) $1+(\sqrt{2}-1)+(3-2\sqrt{2})+\ldots\ldots$to n terms.

(viii) $(a^2-b^2)+(a+b)+\frac{a+b}{a-b}+\ldots\ldots$to n terms.

(ix) $l^p + l^{p+q} + l^{p+2q} + \ldots\ldots$to n terms.

(x) $p(p+q)+p^2(p^2+q^2)+p^3(p^3+q^3)+\ldots\ldots$to r terms. [Split into two series].

(xi) $(n+m)+(n^2+2m)+(n^3+3m)+\ldots\ldots$to p terms.

(xii) $(x+\frac{1}{4})+(3x-\frac{1}{8})+(5x+\frac{1}{16})+\ldots\ldots$to n terms.

12. Sum the infinite series :—

(i) $1+\frac{1}{9}+\frac{1}{97}+\frac{1}{81}+\ldots\ldots$

(ii) $\frac{1}{4}+\frac{1}{2}+\frac{1}{3}+\frac{2}{36}+\ldots\ldots$

(iii) $\frac{4}{5}-\frac{8}{25}+\frac{16}{125}-\frac{32}{625}+\ldots\ldots$

(iv) $\frac{4}{7}+\frac{4}{7^2}+\frac{4}{7^3}+\frac{4}{7^4}+\ldots\ldots$

(v) $\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\frac{1}{9\sqrt{3}}+\ldots\ldots$

(vi) $+32+324+243+\ldots\ldots$

(vii) $9-3+1-\frac{1}{3}+\ldots\ldots$

(viii) $7\frac{1}{2}-5+3\frac{1}{8}-2\frac{2}{8}+\ldots\ldots$

(ix) $(2+\sqrt{3})+1+(2-\sqrt{3})+\ldots\ldots$

(x) $1-3a+9a^2-27a^3+\ldots\ldots$ (when $a<\frac{1}{3}$)

(xi) $\frac{2}{3}+\frac{3}{7}+\frac{4}{15}+\frac{5}{31}+\frac{6}{63}+\frac{7}{127}+\ldots\ldots$ (split into two series.)

(xii) $\frac{a}{x}+\frac{b}{x^2}+\frac{a}{x^3}+\frac{b}{x^4}+\ldots\ldots$(when $x>1$)

13. Without assuming any formula sum the following series in G. P. :—

(i) $2\sqrt{2}+2+\sqrt{2}+1+\ldots\ldots$to 13 terms.

(ii) $\frac{2}{5}+\frac{2}{25}+\frac{2}{125}+\ldots\ldots$to 5 terms.

(iii) $2\frac{1}{4}-1\frac{1}{3}+1-\ldots\ldots$to infinity.

(iv) $\frac{3}{5}+\frac{3}{10}+\frac{3}{20}+\ldots\ldots$*ad infinitum.*

14. Evaluate (*i*) $\cdot 0\dot{3}\dot{5}$; (*ii*) $\cdot 73\dot{2}$; (*iii*) $2\cdot 3\dot{4}\dot{5}$.

15. Find the common ratio of the G. P. whose first term is $\frac{3}{4}$ and sum to infinity is $\frac{9}{4}$.

16. The sum of $1+\sqrt{2}+2+2\sqrt{2}+\ldots$ is $7(\sqrt{2}+1)$; find the number of terms,

17. In a G. P. the sum of the first three terms is 26 and the sum of the first five terms is 242 ; find the series.

18. In an infinite G. P the first term is $\frac{1}{3}$ and the sum is $\frac{1}{2}$; find the common ratio.

19. In an infinite G. P. the first term is 1 and any term equal to the sum of all the succeeding terms. Find the series.

20. Show that the value of $2\cdot\sqrt{2}\cdot\sqrt[4]{2}\cdot\sqrt[8]{2}\cdot\ldots\ldots$ to infinity is 4.

39. Geometric Mean. If there be three quantities in G. P., the middle one is called the *Geometric Mean* between the other two which are the extremes.

If there be a number of quantities in G. P., then all the intermediate ones are called the *Geometric Means* between the first and the last.

G. M. is often written for Geometric mean.

I To find a G. M. between two given quantities.

If a and b be the given quantities, suppose that x is their G. M. Then a, x, b are in G. P.

$$\therefore \frac{x}{a} = \frac{b}{x} \qquad \therefore \quad x = \sqrt{ab}.$$

Thus the G. M. between two quantities is the square root of their product.

II. To insert n Geometric Means between two given quantities.

Let $x_1, x_2, x_3 \ldots x_n$ be the n geometric means between a, b the two given quantities. Then $a + x_1 + x_2 + \ldots + x_n + b$ is a G. P. of $n+2$ terms.

Thus $b = T_{n+2} = ar^{n+1}$, where r is the common ratio.

$$\therefore \qquad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $x_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$; $x_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

$$x_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}} \text{ ; and so on,}$$

The Geometric Means, therefore, are

$$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}},\quad a\left(\frac{b}{a}\right)^{\frac{2}{n+1}},\quad a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},$$

$$\ldots\ldots,\ a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$$

Ex. 1. Insert 5 Geometric Means between 486 and $\frac{2}{3}$.

Since $\frac{2}{3}$ is the 7th term of the G. P. whose first term is 486, we have

$$\frac{2}{3} = T_7 = 486r^6$$

$$\therefore \quad r^6 = \frac{1}{729} = (\pm\tfrac{1}{3})^6.$$

Hence the Geometric means are

$$\pm 486.\tfrac{1}{3} = \pm 162\ ;\quad 486(\tfrac{1}{3})^2 = 54\ ;\quad \pm 486(\tfrac{1}{3})^3 = \pm 18$$
$$486(\tfrac{1}{3})^4 = 6\ ;\quad \pm 486(\tfrac{1}{3})^5 = \pm 2.$$

Note With these Geometric Means there are two series formed :

(1) $486+162+54+18+6+2+\frac{2}{3}$:
(2) $486-162+54-18+6-2+\frac{2}{3}$.

Ex. 2. Find two numbers whose G. M. is 1 and A. M. is $\frac{97}{72}$.

If a and b be the numbers, we have

$$ab = 1, \qquad \text{and} \quad \frac{a+b}{2} = \frac{97}{72},\ i.e.,\ a+b = \frac{97}{36}$$

$$\therefore \qquad a-b = \sqrt{(a+b)^2-4ab} = \sqrt{\frac{9409}{1296}-4} = \frac{65}{36},$$

Thus $a = \frac{9}{4}$ and $b = \frac{4}{9}$.

Exercise XVII

1. Insert 5 Geometric Means between $40\frac{1}{2}$ and $3\frac{5}{9}$.
2. Insert 4 Geometric Means between $\frac{81}{16}$ and $\frac{2}{3}$.
3. Insert 5 Geometric Means between $\frac{2\sqrt{2}}{3\sqrt{3}}$ and $\frac{3\sqrt{3}}{2\sqrt{2}}$.
4. Insert 6 Geometric Means between 12 and $-\frac{8}{405}\frac{64}{729}$.
5. If n Geometric Means be inserted between x and y, show that their product is $(xy)^{\frac{n}{2}}$.

6. If the A. M. between two numbers be twice their G. M., show that the ratio between the numbers is

$$\frac{2+\sqrt{3}}{2-\sqrt{3}}.$$

7. If the A. M. between two numbers be 10 and G. M. be -8, find the numbers.

8. For what value of n will $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ be the G. M. between a and b ?

40. We shall conclude this chapter by solving some miscellaneous examples of useful application.

Ex. 1. Sum $3+33+333+\ldots$ ..to n terms.

If S denotes the required sum we have

$$\begin{aligned}
S &= 3+33+333+\ldots\ldots\text{to } n \text{ terms} \\
&= 3\{1+11+111+\ldots\ldots\text{to } n \text{ terms}\} \\
&= \tfrac{3}{9}\{9+99+999+\ldots\ldots\text{to } n \text{ terms}\} \\
&= \tfrac{1}{3}\{(10-1)+(10^2-1)+(10^3-1)+\ldots\ldots\text{to } n \text{ terms}\} \\
&= \tfrac{1}{3}\{(10+10^2+10^3+\ldots\ldots\text{to } n \text{ terms})-n\} \\
&= \tfrac{1}{3}\left\{\frac{10(10^n-1)}{9}-n\right\} \\
&= \frac{10}{27}(10^n-1)-\frac{n}{3}.
\end{aligned}$$

Ex. 2. Find three numbers in G. P. whose product is 64 and whose sum is $24\frac{4}{5}$.

Suppose that the required numbers are $\frac{a}{r}$, a, ar.

Then their product $a^3=64 \quad \therefore \quad a=4$

and their sum $\frac{a}{r}+a+ar=4\left(r+1+\frac{1}{r}\right)=\frac{124}{5}$

or $$r^2+r+1=\frac{31}{5}r$$

or $$r^2-\frac{26}{5}r+1=0$$

$\therefore$ $$r=5 \text{ or } \tfrac{1}{5}.$$

Hence the numbers are $\frac{4}{5}$, 4, 20.

Ex. 3. If a, b, c, d be in G. P., show that $a^2-b^2, b^2-c^2, c^2-d^2$ are in G. P.

Clearly we have to prove that $(a^2-b^2)(c^2-d^2)=(b^2-c^2)^2$.

Now since a, b, c, d are in G. P., we have

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$\therefore \quad b^2 = ac,\ c^2 = bd$ and $bc = ad$.

Thus $(a^2 - b^2)(c^2 - d^2) = a^2c^2 - b^2c^2 - a^2d^2 + b^2d^2$

$= b^4 - b^2c^2 - b^2c^2 + c^4$

$= (b^2 - c^2)^2$.

Hence the result.

Ex. 4. If x, y, z, be the pth, qth, rth terms of a G. P., show that

$$x^{q-r}y^{r-p}z^{p-q} = 1.$$

Let a be the first term and R the common ratio of the G. P.

Then $\left.\begin{aligned} x &= aR^{p-1} \\ y &= aR^{q-1} \\ z &= aR^{r-1} \end{aligned}\right\} \quad \therefore \quad \begin{aligned} x^{q-r} &= a^{q-r}R^{(q-r)(p-1)} \\ y^{r-p} &= a^{r-p}R^{(r-p)(q-1)} \\ z^{p-q} &= a^{p-q}R^{(p-q)(r-1)} \end{aligned}$

Multiplying these together we get

$$x^{q-r}y^{r-p}z^{p-q} = a^0R^0 = 1.$$

Ex 5 If S_n denotes the sum of n terms of a G. P., find the value of $S_1 + S_2 + S_3 + \ldots\ldots + S_n$.

Let a be the first term and r the common ratio of any G. P.,

then $S_1 = a = \dfrac{a(r-1)}{r-1}$,

$S_2 = \dfrac{a(r^2-1)}{r-1}$,

$S_3 = \dfrac{a(r^3-1)}{r-1}$,

$\ldots\ldots\ldots\ldots$

$S_n = \dfrac{a(r^n-1)}{r-1}$.

$\therefore \quad S_1 + S_2 + S_3 + \ldots\ldots + S_n$

$= \dfrac{a}{r-1}\{(r-1) + (r^2-1) + (r^3-1) + \ldots\ldots + (r^n-1)\}$

$= \dfrac{a}{r-1}\{(r + r^2 + r^3 + \ldots\ldots + r^n) - n\}$

$= \dfrac{a}{r-1}\left\{\dfrac{r(r^n-1)}{r-1} - n\right\}$

$= \dfrac{ar(r^n-1)}{(r-1)^2} - \dfrac{an}{r-1}$

Ex. 6. Sum $(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)$ +.....to n terms.

If S denotes the required sum, then

$$S=\frac{a^2-b^2}{a-b}+\frac{a^3-b^3}{a-b}+\frac{a^4-b^4}{a-b}+\text{.....to } n \text{ terms.}$$

$$=\frac{1}{a-b}\{(a^2+a^3+a^4+\text{.....to } n \text{ terms})$$

$$-(b^2+b^3+b^4+\text{.....to } n \text{ terms})\}$$

$$=\frac{1}{a-b}\left[\frac{a^2(a^n-1)}{a-1}-\frac{b^2(b^n-1)}{b-1}\right]$$

EXERCISE XVIII.

1. Sum to n terms the series whose rth term is

(*i*) $3r+3^r$, (*ii*) 2^r+2r+1, (*iii*) $2^{\frac{1}{r-2}}$.

2. Sum the series :—

(*i*) $5+55+555+$.....to n terms.

(*ii*) $4+44+444+$.....to n terms

(*iii*) $\cdot 7+\cdot 77+\cdot 777+$.....to n terms

(*iv*) $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+$.....to n terms.

[The rth term is $\frac{2^r-1}{2^{r-1}}$].

3. Three numbers in G. P. have their sum 65 and product 3375 ; find them.

4. The sum of 3 numbers in G. P. is 93, and their product is 3375. What are the numbers ?

5. Three numbers in G. P. have their product 512 ; if 8 is added to the first and 6 to the second, they form an A. P. Find the numbers.

6. Three numbers in A. P. have their sum 15 ; increased by 1, 4, 19 respectively they form a G. P. Determine the numbers.

7. If 9 be the sum of 3 numbers in A. P. and if the numbers respectively increased by 3, 7, 16 give a G. P. ; what are the numbers.

8. The product of three numbers in G. P. is 125 and the sum of their products taken two at a time is $87\frac{1}{2}$. Find the numbers.

9. What 4 numbers in A. P. respectively increased

increased by 1, 1, 3, 9 will form a G. P. ?

The numbers $a-3d$, $a-d$, $a+d$, $a+3d$, when increased must give $\frac{a-3d+1}{a-d+1} = \frac{a-d+1}{a+d+3} = \frac{a+d+3}{a+3d+9}$;

and each of these ratios

$$= \frac{(a-3d+1)-2(a-d+1)+(a+d+3)}{(a-d+1)-2(a+d+3)+(a+3d+9)} = \frac{1}{2} \text{ ; etc.}$$

10. Find four numbers in G. P. so that their sum may be 15 and the sum of their squares 85.

[P. U. 1933]

11. If a, b, c, d are in G. P., show that

(*i*) $(a-b)^2, (b-c)^2, (c-d)^2$ are in G. P.

(*ii*) $(a^2+b^2+c^2), (ab+bc+cd), (b^2+c^2+d^2)$ are in G. P.

(*iii*) $(a+b)^2, (b+c)^2, (c+d)^2$ are in G. P.

(*iv*) $\frac{1}{(a-b)^2}, \frac{1}{(b-c)^2}, \frac{1}{(c-d)^2}$ are in G. P.

(*v*) $(b+c)(b+d)=(c+a)(c+d)$

(*vi*) $(a+b)^2+2(b+c)^2+(c+d)^2=(a+b+c+d)^2$

(*vii*) $\frac{1}{(a+b)^2}, \frac{1}{(b+c)^2}, \frac{1}{(c+d)^2}$ are in G P.

12. If a, b, c be in G. P., show that

(*i*) $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3} = \frac{a}{b^2c^2}+\frac{b}{c^2b^2}+\frac{c}{a^2b^2}$.

(*ii*) $\frac{1}{a^2-b^2}+\frac{1}{b^2} = \frac{1}{b^2-c^2}$.

(*iii*) $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$.

13. What must be added to the numbers a, b, c, so that the new numbers may be in G. P ?

14. In a G. P. the $(m+n)$th term is p and $(m-n)$th term is q ; show that mth term is $\sqrt{pq}$.

15. If $y=x+x^2+x^3+\ldots\ldots$to ∞, x being <1 ; show that $x=\frac{y}{1+y}$.

16. Find the common ratio of an infinite G. P. in which any term is $\frac{2}{3}$ of the sum of all the terms that follow it.

17. If a, b, c be in A. P. and x, y, z in G. P., then prove that $x^by^cz^a=x^cy^az^b$.

18. If the pth, qth, rth terms of a G. P. be in G. P., then p, q, r are in A. P.

19. If P denotes the product of n terms of a G. P., S denotes their sum and R, the sum of their reciprocals, prove that $P=\left(\frac{S}{R}\right)^{\frac{n}{2}}$.

20. If S_1, S_2, S_3 be the sums to n, $2n$, $3n$ terms of a G. P., prove that

(*i*) $(S_1-S_2)^2=S_1(S_3-S_2)$

(*ii*) $S_1^2+S_2^2=S_1(S_2+S_3)$.

21. If $S_1, S_2, S_3, \ldots\ldots S_n$ be the the sums of n infinite Geometric series whose first terms are 1, 2, 3,.....n, and common ratio $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \frac{1}{n+1}$ respectively, then show that $S_1+S_2+S_3+\ldots\ldots+S_n=\frac{1}{2}n(n+3)$.

22. If $S_1, S_2, S_3, \ldots\ldots S_n$ be the sums of n infinite Geometric series, the first term of each being unity and the common ratios $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r_3}, \ldots\ldots \frac{1}{r^n}$ respectively, then show that $\frac{1}{S_1}+\frac{1}{S_2}+\frac{1}{S_3}\ldots\ldots+\frac{1}{S^n}=n-\frac{r^n-1}{r^n(r-1)}$.

23 The first term of a convergent Geometric series is x and the sum to infinity is y. Find the sum of n terms.

24. If x, y, z be in G. P. and if a, b be the Arithmetic means between x, y and y, z respectively, then prove that $\frac{2}{y}=\frac{1}{a}+\frac{1}{b}$ and $\frac{x}{a}+\frac{z}{b}=2$.

25. A tennis ball when hit with a racket strikes the ground at a distance of x ft. It rebounds and again strikes at a distance $\frac{2}{3}$ of the former ; and so on. Find the distance from the racket at which it will come to rest.

CHAPTER VII

Harmonic Progressions

41. Definition. A type of *series* or sequence in which the successive terms are the recipocals of the corresponding terms of some Arithmetical Progression, is called the Harmonical Progression.

The terms of the series (1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots$

(2) $\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\ldots\ldots\ldots$

(3) $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots\ldots$

are in Harmonic Progression since, obviously, the series formed by the reciprocals of their terms are in A P.

Harmonic Progression is briefly written as H. P.

42. Condition for any three quantities to be in H. P. To find the condition that the quantities a, b, c be in H.P., it is obvious form the definition that $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ should be in A. P.

Now these quantities are in A. P. if

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$$

or $$\frac{a-b}{ab}=\frac{b-c}{bc}$$

or $$\frac{a-b}{b-c}=\frac{a}{c}.$$

Cor. Retracing the steps of the above process it can be shown that if $\frac{a-b}{b-c}=\frac{a}{c}$, then a, b, c are in H. P.

This leads to another definition of a Hormonic Progression. A series in which any three terms a, b, c satisfy the relation $\frac{a-b}{b-c}=\frac{a}{c}$, is called the Harmonic Progression.

Ex. 1. First two terms of an H. P. are $\frac{2}{5}$ and $\frac{2}{7}$; find the third term.

Let c be the third term. Then since $\frac{2}{5}$, $\frac{2}{7}$, c are in H. P.,

we have $\frac{\frac{2}{5}-\frac{3}{4}}{\frac{3}{4}-c}=\frac{\frac{2}{5}}{c}$ or $\frac{7}{20}c=\frac{2}{5}(c-\frac{3}{4})$

whence $c=6$.

Ex. 2. Find the 11th term of the series $\frac{2}{5}, \frac{8}{17}, \frac{1}{11}, \cdots$

The series $\frac{5}{2}, \frac{17}{8}, \frac{11}{1}$.....formed by the reciprocals of the given terms is evidently an A. P. with common difference $\frac{5}{8}$.

Now the 11th term of the A. P. $=\frac{3}{2}+10\times\frac{5}{8}=\frac{31}{4}$.

$\therefore$ the 11th term of the given H. P. $=\frac{4}{31}$.

43. Harmonic Mean. When three numbers are in H. P. the middle one is called the *Harmonic Mean* between the other two.

If a number of quantities be in H. P., all the intermediate ones are called the *Harmonic Means* between the first and the last.

Harmonic Mean is briefly written as H. M.

I. *To find an H. M. between two given quantities.*

Let x be the H. M. between the two given quantities a and b. Then a, x, b are in H. P.

$$\therefore \quad \frac{a-x}{x-b}=\frac{a}{b} \text{ or } b(a-x)=a(x-b)$$

$$\text{or} \quad x(a+b)=2ab, \qquad \therefore \quad x=\frac{2ab}{a+b}.$$

Thus the H. M. between two quantities is twice their product divided by their sum.

II. *To insert n Harmonic Means between two given quantities.*

Let the given quantities be a and b, and suppose that the n Harmonic Means are $x_1, x_2, x_3, \cdots\cdots x_n$.

Then $\frac{1}{a}, \frac{1}{x_1}, \frac{1}{x_2}, \cdots\cdots \frac{1}{x_n}, \frac{1}{b}$ is an A. P. of $n+2$ terms. If d be the common difference of this A. P., then

$$\frac{1}{b}=\frac{1}{a}+(n+1)d$$

whence $$d=\frac{\frac{1}{b}-\frac{1}{a}}{n+1}=\frac{a-b}{ab(n+1)}.$$

Thus $\frac{1}{x_1} = \frac{1}{a} + \frac{a-b}{ab(n+1)}$; $\frac{1}{x_2} = \frac{1}{a} + \frac{2(a-b)}{ab(n+1)}$;

$\frac{1}{x_3} = \frac{1}{a} + \frac{3(a-b)}{ab(n+1)}$; ... $\frac{1}{x_n} = \frac{1}{a} + \frac{n(a-b)}{ab(n+1)}$.

Hence the required Harmonic Means are

$$\frac{(n+1)ab}{nb+a}, \frac{(n+1)ab}{(n-1)b+2a}, \ldots\ldots, \frac{(n+1)ab}{b+na}$$

Ex. Insert 6 Harmonic Means between $\frac{3}{7^6}$ and $\frac{1}{2}$.

Here we have 2 as the 8th term of the A. P. whose first term is $\frac{7}{3^6}$. Thus d the common difference $= \frac{1}{7}(2 - \frac{7}{3^6}) = -\frac{10}{3^6}$.

$\therefore$ the Arithmetic means between $\frac{7}{3^6}$ and 2 are

$$22, \frac{5}{3^6}, \frac{4}{3^6}, 12, \frac{2}{3^6}, \frac{1}{3^6}.$$

Hence the required Harmonic means are

$$\frac{1}{22}, \frac{3}{56}, \frac{3}{46}, \frac{1}{12}, \frac{3}{26}, \frac{3}{16}.$$

44. Now we shall solve some Miscellaneous examples on H. P. The first two of them are important.

Ex. 1. *Three quantities a, b, c are in A. P., G. P., or H. P. according as*

$$\frac{a-b}{b-c} = \frac{a}{a} \text{ or } \frac{a}{b} \text{ or } \frac{a}{c}.$$

First case. When $\frac{a-b}{b-c} = \frac{a}{a}$, we have

$$a-b = b-c$$

or $$a+c = 2b$$

Hence a, b, c are in A. P.

Second case When $\frac{a-b}{b-c} = \frac{a}{b}$, we have

$$ab - b^2 = ab - ac, \text{ or } ac = b^2$$

Hence a, b, c are in G. P.

Third case. When $\frac{a-b}{b-c} = \frac{a}{c}$ we have

$$\frac{a-b}{a} = \frac{b-c}{c} \text{ or } \frac{a-b}{ab} = \frac{b-c}{bc}$$

or $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Hence a, b, c are in H. P.

Ex. 2. *If A, G, and H be the Arithmetic, the*

Geometric and the Harmonic means between any two positive quantities, then

(*i*) *A, G, H. form a G. P.*

and (*ii*) $A > G > H$.

Let a and b be the two quantities. Then

$$A = \frac{a+b}{2}\ ;\ G = \sqrt{ab}\ ;\ H = \frac{2ab}{a+b}\ .$$

(*i*) Now $A.\ H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2$.

Hence A, G, H are in G. P.

(*ii*) $A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a+b-2\sqrt{ab}$

$$= \frac{1}{2}\ (\sqrt{a} - \sqrt{b})^2$$

which is positive, and therefore $A > G$.

$$\text{Next } G - H = \sqrt{ab} - \frac{2ab}{a+b} = \frac{\sqrt{ab}(a+b-2\sqrt{ab})}{a+b}$$

$$= \frac{\sqrt{ab}(\sqrt{a}-\sqrt{b})^2}{a+b}$$

which is positive, and therefore $G > H$.

Hence $A > G > H$.

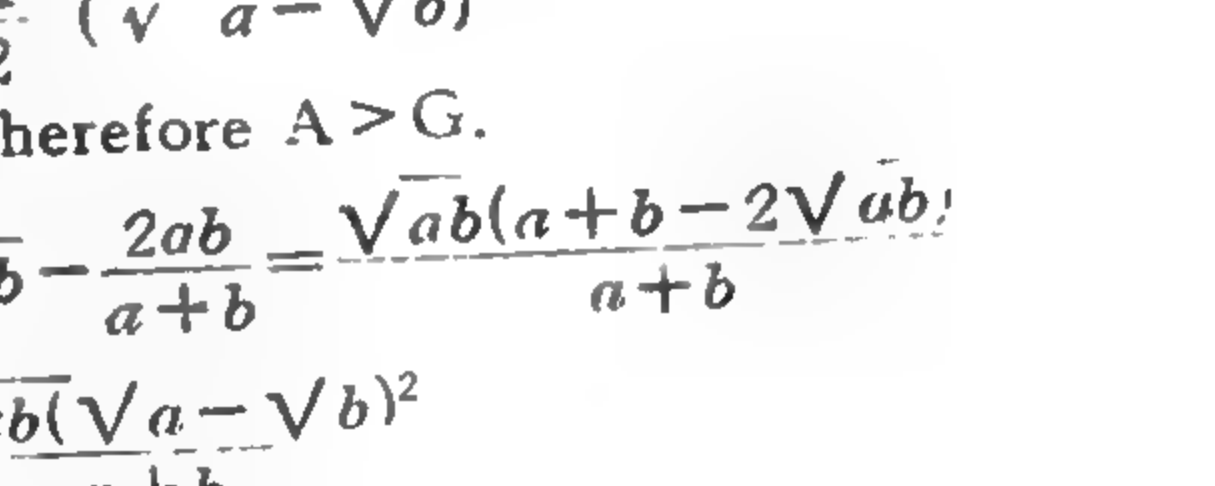
Hence A, G, H between any two positive *unequal numbers* form a G. P. in descending order of magnitude.

Question : How will the proof of (*ii*) above be affected, if a and b are not positive ?

Ex. 3. Representation of A, G and H geometrially.

Let AB and AC represent the two quantities a and b.

Draw a semi-circle on BC as diameter, and draw AT a tangent to the semi-circle as shown in the figure. Join OT, O being the middle point of BC and draw TN perpendicular to BC.



Now, $A = \frac{1}{2}(a+b)$

$= \frac{1}{2}(AB+AC)$

$$= \tfrac{1}{2}[AO - BO + AO + OC]$$
$$= AO, \text{ since } [BO = OC]$$
$$G = \sqrt{ab}$$
$$= \sqrt{AB.AC}$$
$$= \sqrt{AT^2} = AT.$$

[tangent and segments of a secant.

$$H = \frac{2ab}{a+b}$$
$$= \frac{2.AB\ AC}{AB+AC}$$
$$= \frac{AB\ AC}{\frac{1}{2}(AB+AC)}$$
$$= \frac{AT^2}{AO}$$
$$= AN,$$

Since from the right-angled triangle ATO, AN.AO = AT².

Hence, A, G, H are respectively represented by the lengths AO, AT and AN.

Ex. 4. If a, b, c be in H. P., show that $a^2 + c^2 > 2b^2$.

We are given that b is the H. M. between a and c.

Since the G. M. is greater than the H. M., we have

$$\sqrt{ac} > b, \quad i.e., \quad ac > b^2.$$

Now $a^2 + c^2 - 2ac$ is a positive quantity;

$$\therefore \quad a^2 + c^2 > 2.ac$$

Hence $a^2 + c^2 > 2b^2$.

Ex. 5. If a, b, c be in H. P., show that

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H. P.

Since a, b, c are in H. P., we have

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A. P.

i. e., $\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ in A. P.

i. e., $1+\frac{b+c}{a}, 1+\frac{c+a}{b}, 1+\frac{a+b}{c}$ in A. P.

i. e., $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ in A. P.

Hence $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H. P.

Ex. 6. If a, b, c lie in G. P. and if $a^x=b^y=c^z$, show that x, y, z are in H. P.

Here $a^x=b^y=c^z$ give $a=b^{\frac{y}{x}}$ and $c=b^{\frac{y}{z}}$

$$\therefore \quad ac=b^{\frac{y}{x}+\frac{y}{z}}.$$

But a, b, c are in G. P.; *i. e.*, $ac=b^2$

Thus $b^2=b^{y\left(\frac{1}{x}+\frac{1}{z}\right)}$ $\quad\therefore\quad 2=y\left(\frac{1}{x}+\frac{1}{z}\right)$

$$\therefore \quad \frac{1}{y}-\frac{1}{x}=\frac{1}{z}-\frac{1}{y}.$$

Hence x, y, z are in H. P.

Ex. 7. An A. P. and an H. P. have the same first term, the same last term and the same number of terms; prove that the product of the rth term from the beginning in one series and the rth term from the end in the other is independent of r.

Let a be the first term, and l the last term, and n the number of terms in each series.

Let d be the common difference of the A. P., and d' that of the arithmetic progression corresponding to the given H. P. Then we must have

$$l=a+(n-1)d, \text{ and } \frac{1}{l}=\frac{1}{a}+(n-1)d';$$

whence $d=\frac{l-a}{n-1}$, and $d'=\frac{a-l}{al(n-1)}$

$$\therefore \quad d'=-\frac{d}{al} \qquad \ldots\ldots\ldots\ldots(1).$$

Let t be the rth term of the A. P. from the beginning, and t' the rth term of the H. P. from the end. Then

$$t=a+(r-1)d,$$

$$\text{and } t'=\frac{1}{l}-(r-1)d'$$

$$=\frac{1}{l}+\frac{(r-1)d}{al} \quad \text{[from (1)]}$$

$$=\frac{1}{al}\{a+(r-1).d\}$$

$$= \frac{1}{al} \cdot t$$

Hence $tt' = al$, which is independent of r.

EXERCISE XIX

1. Find the first term of an H. P. whose second and third terms are $4\frac{2}{7}$ and $4\frac{6}{13}$ respectively.

2. What is the second term of the H. P. whose first and third terms are $\frac{1}{3}$ and $\frac{3}{26}$ respectively ?

3. What is the H. M. between $3\frac{1}{3}$ and $1\frac{3}{4}$?

4. Insert 3 Harmonic means between 4 and $\frac{1}{4}$.

5. Find 5 Harmonic means between $\frac{1}{6}$ and $\frac{1}{24}$.

6. In the series $\frac{2}{9}$, $\frac{1}{4}$, $\frac{2}{7}$...find the 6th term.

7. What is the 20th term of the series $\frac{3}{7}$, $\frac{3}{5}$, 1,... ?

8. Find the nth term of the series 1, 4, -2, $-\frac{4}{5}$,...

9. The third term of an H. P. is $\frac{3}{8}$ and the 13th term is $\frac{4}{25}$; find the 7th term.

10. Continue the series $1\frac{2}{7} + 1\frac{4}{9} + \frac{1}{4} +$...to 3 terms each way.

11. Between -3 and 3 insert two numbers such that the first three may be in H. P. and the last three in G. P.

12. If $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ be the H. M. between a and b ; find n.

13. The sum of three numbers in H. P. is $\frac{37}{15}$; the first of them is $\frac{1}{5}$; find the others.

14. The G. M. between two numbers is 24 and their H. M. is $14\frac{2}{5}$; find the numbers.

15. The A. M. between two numbers exceeds their G. M. by $\frac{3}{5}$; and the G. M. exceeds their H. M. by $\frac{9}{5}$; find the numbers.

16. If the mth term of an H. P. be n and the nth term be m, find the $(m+n)$th term.

17. If a, b, c be in H. P., show that

$$\frac{1}{a} + \frac{1}{a-b} + \frac{1}{c} + \frac{1}{c-b} = 0.$$

18. If a^2, b^2, c^2 be in A. P., show that $b+c$, $c+a$, $a+b$ are in H. P.

19. If a, b, c be in G. P., show that

$$\frac{a+b}{2}, b, \frac{b+c}{2} \text{ are in H. P.}$$

20. If a, b, c be in H. P., show that

(*i*) $\frac{1}{a}+\frac{1}{b+c}, \frac{1}{b}+\frac{1}{c+a}, \frac{1}{c}+\frac{1}{a+b}$ are in H. P.

(*ii*) $\frac{1}{a(b+c)}, \frac{1}{b(c+a)}, \frac{1}{c(a+b)}$ are in H. P.

21. If a, b, c be respectively the p*th*, q*th* and r*th* terms of an H. P., prove that

$$(q-r)bc+(r-p)ca+(p-q)ab=0.$$

22. Prove that the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ has equal roots, if a, b, c are in H. P.

23. If a, b, c be in H. P., prove that

(*i*) $\frac{1}{b-a}+\frac{1}{b-c}=\frac{2}{b}$.

(*ii*) $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2.$

24. If a, b, c be in A. P. and if p, q, r be the Geometric Means between ca, ab ; ab, bc : bc, ca, then prove that

(*i*) p^2, q^2, r^2 are in A. P.

(*ii*) $q+r, r+p, p+q$ are in H. P.

25. Show that x, y, z are in H. P., if

$\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ and if p, q, r are in A. P.

26. If a, b, c be in A. P., and if b, c, a are in H. P., show that c, a, b are in G. P.

27. If a, b, c be in A. P., ; $\gamma a, b\beta, c\Upsilon$ in G. P. ; and α, β, Υ in H. P , then

$$\frac{a}{\Upsilon}+\frac{\Upsilon}{a}=\frac{a}{c}+\frac{c}{a}.$$

28. If a, b, c, d be in H. P., show that $a+d>b+c$.

29. Given that a, b, c are in G. P, that p is the A. M. between a and b, and that q is the A. M. between b and c ; show that b is the H. M. between p and q.

30. If a, b, c be in A. P. show that

$$\frac{bc}{a(b+c)}, \frac{ca}{b(c+a)}, \frac{ab}{c(a+b)} \text{ are in H. P.}$$

31. In the figure of Ex. 3, Art. 44 show that $A>G>H$.

32. If the terms of an H. P. be multiplied with, or divided by any the same number throughout, the resulting terms also are in H. P.

CHAPTER VIII

Some Other Series

45. We shall now consider the summation of some such series as do not fall under any of the foregoing heads. Such series will admit of simple methods of summation.

From what has been said about the three Progressions, it is clear that a series can be summed up easily when the law of formation of its successive terms is known. In the summation of any series, therefore, our efforts will generally be to establish the law. It will be noticed that in most cases the law is established by finding the nth term ; or the *general term* as it is called.

46 Before proceeding with the proper subject of summation it will be useful to explain the notation Σ (sigma). This Greek letter Σ when placed before the nth term signifies the sum of all terms got by giving all positive integral values to n. Thus

Σ_n signifies $1+2+3+\ldots\ldots+n$;

$\Sigma\frac{a^n}{n}$ signifies $\frac{a}{1}+\frac{a^2}{2}+\frac{a^3}{3}+\ldots\ldots+\frac{a^n}{n}$;

Σn^n signifies $1+2^2+3^3+4^4+\ldots\ldots+n^n$

$\sum_{n=1}^{n=10}(1+2n)=3+5+7+\ldots\ldots+21$

It should be noted that Σ does not indicate the actual value of the sum of a number of terms as S does.

Two Important Theorems on Σ.

(1) To show that $\Sigma af(n)=a\Sigma f(n)$, where $f(n)$ represents a function of n, and a is a co-efficient independent of n.

Here, $T_n=af(n)$ so that

$T_1=af(1)$,

$T_2=af(2)$,

. . . . . .

$T_{n-1}=af(n-1)$

$T_n=af(n)$.

$\therefore \quad \Sigma af(n)=T_1+T_2+\ldots+T_{n-1}+T_n$

$=a[f(1)+f(2)+\ldots+f(n-1)+f(n)]$

$=a\Sigma f(n)$.

Example. Show that $\Sigma 2(n^2) = 2\Sigma n^2$

Here, $T_n = 2.n^2$
$T_1 = 2.1^2$
$T_2 = 2.2^2$
. . . . . .
$T_{n-1} = 2(n-1)^2$
$T_n = 2.n^2$.

$\therefore \quad \Sigma 2(n^2) = T_1 + T_2 + \ldots + T_{n-1} + T_n$
$= 2[1^2 + 2^2 + \ldots + (n-1)^2 + n^2]$
$= 2\Sigma n^2$.

(2) To show that $\Sigma\{f(n) + F(n)\}$
$= \Sigma f(n) + \Sigma F(n)$,
where $f(n)$ and $F(n)$ are functions of n

Here $T_n = f(n) + F(n)$
$\therefore \quad T_1 = f(1) + F(1)$
$T_2 = f(2) + F(2)$
. . . . . .
$T_{n-1} = f(n-1) + F(n-1)$
$T_n = f(n) + F(n)$

$\therefore \quad \Sigma\{f(n) + F(n)\} = T_1 + T_2 + \ldots + T_{n-1} + T_n$
$= [f(1) + f(2) + \ldots + f(n-1) + f(n)]$
$+ [F(1) + F(2) + \ldots + F(n-1) + F(n)]$
$= \Sigma f(n) + \Sigma F(n)$.

Ex. Show that $\Sigma(n^2 + n) = \Sigma n^2 + \Sigma(n)$.

47. Arithmetico Geometric Series. A type of series in which each term is the product of the corresponding terms of an A. P. and a G. P. is called the *Arithmetico Geometric Series*

For example the A. P.

$$a + (a+d) + (a+2d) + \ldots\ldots + (a + n - 1d)$$

and the G. P.

$$1 + r + r^2 + \ldots\ldots + r^{n-1}$$

will give the Arithmetico-Geometric series

$$a + (a+d)r + (a+2d)r^2 + \ldots\ldots + (a + n - 1d)r^{n-1}.$$

Since the law of formation is evident, we shall at once establish the method

To find the sum of an Arithmetico-Geometric series upto n terms.

Suppose S is the required sum of the above series. Then

$$S = a + (a+d)r + (a+2d)r^2 + \ldots\ldots + (a+n-1d)r^{n-1}$$

$$\therefore\ rS = ar + (a+d)r^2 + \ldots + (a+\overline{n-2}d)r^{n-1} + (a+n-1d)r^n.$$

Subtracting

$$(1-r)S = a + dr + dr^2 + \ldots\ldots + dr^{n-1} - (a+n-1d)r^n$$

which, except the first and the last terms, is a G. P. of $n-1$ terms.

$$\therefore\quad (1-r)S = a + \frac{dr(1-r^{n-1})}{1-r} - (a+n-1d)r^n.$$

$$\therefore\quad S = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+n-1d)r^n}{1-r}.$$

Cor. If $r<1$ and the above series is infinite, then the sum is given by $S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$, since the limit of $\frac{(a+\overline{n-1}d)r^n}{1-r}$ is zero when n approches infinity, r being less than 1.

Ex. 1. Sum to n terms and to infinity the series

$$1 + 5x + 9x^2 + 13x^3 + \ldots\ldots$$

where $x<1$.

Let S_n denote the sum to n terms and S_∞ the sum to infinity.

Then $\quad S_n = 1 + (1+4)x + (1+2.4)x^2 + \ldots + (1+n-1.4)x^{n-1}$

$$\therefore\quad x\,S_n = \qquad x + (1+4)x^2 + \ldots\ldots$$
$$+ (1+\overline{n-2}.4)x^{n-1} + (1+n-1.4)x^n$$

$$\therefore\quad (1-x)S_n = 1 + 4x + 4x^2 + \ldots + 4x^{n-1} - (4n-3)x^n$$
$$= 1 + \frac{4x(1-x^{n-1})}{(1-x)} - (4n-3)x^n$$

$$\therefore\quad S_n = \frac{1}{1-x} + \frac{4x(1-x^{n-1})}{(1-x)^2} - \frac{(4x-3)x^n}{1-x}.$$

Evidently $S_\infty = \frac{1}{1-x} + \frac{4x}{(1-x)^2}$.

Ex. 2. Sum to infinity the series

$$1 - \frac{4}{5} + \frac{7}{5^2} - \frac{10}{5^3} + \ldots\ldots$$

Let S be the required sum. Then

$$S = 1 - \frac{4}{5} + \frac{7}{5^2} - \frac{10}{5^3} + \ldots\ldots \text{to } \infty$$

$$-\frac{1}{5}S = -\frac{1}{5} + \frac{4}{5^2} - \frac{7}{5^3} + \ldots\ldots \text{to } \infty$$

$$\therefore \quad (1+\frac{1}{5})S = 1 - \frac{3}{5} + \frac{3}{5^2} - \frac{3}{5^3} + \ldots\ldots \text{to } \infty$$

$$= 1 - \frac{\frac{3}{5}}{1+\frac{1}{5}},$$

$$\therefore \quad S = \frac{5}{12}.$$

Ex. 3. Find the sum to n terms of the series

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots\ldots$$

Let S denote the sum to n terms.

Then $S = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\ldots\ldots + n^2x^{n-1}$

$$xS = \qquad x + 2^2x^2 + 3^2x^3 + \ldots\ldots + (n-1)^2x^{n-1} + n^2x^n$$

$$\therefore \quad (1-x)S = 1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots + (2n-1)x^{n-1} - n^2x^n \ldots\ldots(1)$$

Again, multiplying by x,

$$x(1-x)S = \qquad x + 3x^2 + 5x^3 + \ldots\ldots\ldots + (2n-1)x^{n-1} + (2n-1)x^n - n^2x^{n+1} \ldots\ldots(2)$$

Subtracting (2) from (1) we get

$$(1-x)^2S = 1 + 2x + 2x^2 + 2x^3 + \ldots + 2x^{n-1} - (n^2+2n-1)x^n + n^2x^{n+1}$$

$$= 1 + \frac{2x(1-x^{n-1})}{1-x} - (n^2+2n-1)x^n + n^2x^{n+1}$$

$$\therefore \quad S = \frac{1}{(1-x)^2} + \frac{(1-x^{n-1})2x}{(1-x)^3} - \frac{(n^2+2n-1)x^n}{(1-x)^2} + \frac{n^2x^{n+1}}{(1-x)^2}$$

Note.—Evidently the sum to infinity is

$$\frac{1}{(1-x)^2}+\frac{2x}{(1-x)^3}=\frac{1+x}{(1-x)^3}\ .$$

EXERCISE XX.

1. Sum $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+$.....to n terms.

2. Sum $1+\frac{4}{5}+\frac{7}{25}+\frac{10}{125}+$...to ∞.

3. Sum $1-\frac{5}{6}+\frac{9}{6^2}-\frac{13}{6^3}+$...to ∞.

4. Sum $1+1+\frac{3}{4}+\frac{1}{2}+\frac{5}{16}+$...to n terms.

5. Sum $\frac{2}{3}-\frac{5}{9}+\frac{8}{27}-\frac{11}{81}+$.....to ∞.

6. Sum $1+2.2+3.4+4.8+$.....to n terms.

7. Sum $1+\frac{1}{5}+\frac{7}{25}+\frac{10}{125}+$.....to n terms and hence to ∞.

8. Sum $1+4x+7x^2+10x^3+$...to n terms.

9. Sum $1+3x^2+5x^4+7x^6+$...to n terms and hence to ∞ when $x<1$.

10. Sum $1-x+2x^2-3x^3+$...to ∞, $x<1$.

11. Sum $1+3^2x+5^2x^2+7^2x^3+$...to ∞, $x<1$.

12. Sum $\frac{5}{6}+\frac{7}{16}+\frac{9}{54}+\frac{11}{162}+$...to ∞.

13. Sum $n+(n-1)x+(n-2)x^2+$.....to n terms.

14. Sum $1+(1+a)r+(1+a+a^2)r^2+(1+a+a^2+a^3)r^3$ +...to ∞, where a and r are proper fractions.

15. Sum $1+\frac{x}{a}(1+x)+\frac{x^2}{a^2}(1+x+x^2)+\frac{x^3}{a^3}(1+x+x^2+x^3)+$...to ∞, where $a>1$ and $x<1$.

16. Show that $3^{\frac{1}{4}}.9^{\frac{1}{8}}.27^{\frac{1}{16}}.81^{\frac{1}{32}}$...to $\infty=3$.

48 Natural Numbers.

I. *To find the sum of the first n natural numbers.*

Let S denote the sum $1+2+3+4+$.. ..$+n$, so that $S=\Sigma n$.

It is an A. P. with first term and common difference each equal to unity.

$$\therefore\quad S=\frac{n}{2}\left\{2\times 1+(n-1)\times 1\right\}$$

$$=\frac{n(n+1)}{2}.$$

Note.—Using the Σ notation, we can write this formula as

$$\Sigma n = \frac{n(n+1)}{2}.$$

Ex. $\sum_{n=1}^{n=9} n = \frac{9\times 10}{2} = 45.$

II. *To find the sum of the squares of the first n natural numbers.*

Let S denote the sum $1^2+2^2+3^2+\ldots+n^2$.

Since, we know that for all values of x

$$x^3-(x-1)^3 \equiv 3x^2-3x+1,$$

giving to x values 1, 2, 3,..... $n-1$, n successively, we get

$$1^3-0^3 = 3.1^2-3\cdot 1+1$$
$$2^3-1^3 = 3.2^2-3\cdot 2+1$$
$$3^3-2^3 = 3.3^2-3\cdot 3+1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n-1)^3-(n-2)^3 = 3(n-1)^2-3(n-1)+1$$
$$n^3-(n-1)^3 = 3n^2-3n+1.$$

Adding up both the sides of these equalities, column by column we have

$$n^3 = 3[1^2+2^2+3^2+\ldots+n^2]-3[1+2+3+\ldots+n]+n$$

or $$n^3 = 3S-3\,\frac{n(n+1)}{2}+n$$

$$\therefore \quad 3S = n^3-n+\tfrac{3}{2}n(n+1)$$
$$= n(n+1)\{(n-1)+\tfrac{3}{2}\}$$
$$= \tfrac{1}{2}n(n+1)(2n+1)$$
$$\therefore \quad S = \tfrac{1}{6}n(n+1)(2n+1).$$

Note.—Using the Σ notation we can write this formula as

$$\Sigma n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Ex. $\sum_{n=1}^{n=20} n^2 = \tfrac{1}{6}\,20(20+1)(40+1) = 2870$

III. *To find the sum of the cubes of the first n natural numbers.*

Let S denote the sum $1^3+2^3+3^3+\ldots+n^3$.

Since for all values of x,

$$x^4-(x-1)^4 \equiv 4x^3-6x^2+4x-1,$$

giving to x values 1, 2, 3.. $n-1$, n successively, we get

$$1^4-0^4=4.1^3-6.1^2+4.1-1$$
$$2^4-1^4=4.2^3-6.2^2+4.2-1$$
$$3^4-2^4=4.3^3-6.3^2+4.3-1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n-1)^4-(n-2)^4=4(n-1)^3-6(n-1)^2+4(n-1)-1$$
$$n^4-(n-1)^4=4n^3-6n^2+4n-1.$$

Adding up both the sides of these equalities column by column, we have

$$n^4=4[1^3+2^3+3^3+\ldots+n^3]-6[1^2+2^2+3^2+\ldots+n^2]+4[1+2+3+\ldots+n]-n$$

or $$n^4=4S-6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}-n$$

$$\therefore \quad 4S=n^4+n+n(n+1)(2n+1)-2n(n+1)$$
$$=n(n+1)\{(n^2-n+1)+(2n+1)-2\}$$
$$=n(n+1)(n^2+n)$$
$$=n^2(n+1)^2.$$

$$\therefore \quad S=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Note.—Using the Σ notation we can write this formula as

$$\Sigma n^3=\left\{\frac{n(n+1)}{2}\right\}^2=(\Sigma n)^2.$$

Notice that *the sum of the cubes of the first n natural numbers is the square of the sum of the first n natural numbers.*

Caution.—$(\Sigma n)^2$ is not the same thing as Σn^2.

Ex. Find the sum of n terms of the series whose nth term is an^3+bn^2+cn+d

Since $T_n=an^3+bn^2+cn+d$,

$\therefore$ $T_1=a.1^3+b.1^2+c.1+d$

$T_2=a.2^3+b.2^2+c.2+d$

$T_3=a.3^3+b.3^2+c.3+d$

. . . . . .

$T_{n-1}=a(n-1)^3+b(n-1)^2+c(n-1)+d$

and $T_n=an^3+bn^2+cn+d.$

Thus if S denotes the required sum, we have

$$S=T_1+T_2+T_3+\ldots\ldots+T_{n-1}+T_n$$
$$=a(1^3+2^3+3^3+\ldots+n^3)+b(1^2+2^2+3^2+\ldots+n^2)+c(1+2+3+\ldots+n)+dn$$

$$= a\left\{\frac{n(n+1)}{2}\right\}^2 + b\frac{n(n+1)(2n+1)}{6}$$

$$+ c\frac{n(n+1)}{2} + dn.$$

Note. Using the Σ notation, this equation can be solved thus :-

$$S = \Sigma T_n = \Sigma(an^3 + bn^2 + cn + d)$$
$$= a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + dn$$
$$= a\left\{\frac{n(n+1)}{2}\right\}^2 + b\frac{n(n+1)(2n+1)}{6}$$

$$+ c\frac{n(n+1)}{2} + dn.$$

49. Method of Differences. Often it is not easy to find the general term, and therefore, the sum of a series, although the law of formation of the series may be recognised. In some cases it becomes helpful to frame another series from the given series. Thus if T_1, T_2, T_3...be the terms of the given series, then the new series generally framed has its terms $(T_2 - T_1)$, $(T_3 - T_2)$, $(T_4 - T_3)$....This new series may belong to a type of series whose general term T_n can be easily formed and therefore, in general, the sum of the original series may be calculated.

The method will be clear from the following examples

Ex. 1. Sum to n terms the series

$$28 + 50 + 78 + 112 + \ldots\ldots$$

If S is the required sum, then

$$S = 28 + 50 + 78 + 112 + \ldots\ldots + T_n$$

Also $S = \quad 28 + 50 + 78 + \quad + T_{n-1} + T_n$

Subtracting, $0 = 28 + 22 + 28 + 34 + \ldots\ldots - T_n$.

On the right-hand side there is a series of $n+1$ terms Excluding the first and the last terms, the remaining $n-1$ terms form an A. P. Hence

$$T_n = 28 + \frac{n-1}{2}\{2.22 + (n-2)6\}$$
$$= 28 + 22(n-1) + 3(n-1)(n-2)$$
$$= 3n^2 + 13n + 12.$$

Now $S=\Sigma T_n=\Sigma(3n^2+13n+12)$
$=3\Sigma n^2+13\Sigma n+12n$ [Art. 44 Theor. 1 & 2]
$=\frac{n(n+1)(2n+1)}{2}+\frac{13n(n+1)}{2}+12n$
$=n^3+15n^2+12n.$

Ex. 2. Sum the series $3+33+333+$.....to n terms.
Let S be the required sum. Then
$S=3+33+333+3333+.....+T_n$
Also $S=\quad 3+\ 33+\ 333+...+T_{n-1}+T_n$
$\therefore\quad 0=3+30+300+3000.....-T_n$
$\therefore\quad T_n=3+30+300.....$to n terms
$=\frac{3(10^n-1)}{10-1}=\frac{1}{3}(10^n-1).$

Hence $S=\Sigma T_n=\Sigma\frac{1}{3}(10^n-1)$
$=\frac{1}{3}\{\Sigma 10^n-n\}$
$=\frac{1}{3}\{(10+10^2+10^3+.....+10^n)-n\}$
$=\frac{1}{3}\left\{\frac{10(10^n-1)}{9}-n\right\}$
$=\frac{10}{27}(10^n-1)-\frac{n}{3}$.

50. Mathematical Induction. There are many important formulæ which it is not easy to establish directly. Yet they are true. Their validity is ascertained by what is called the process of *Induction*. The method will be best explained and illustrated by an example.

Ex. Prove that the sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

Assume that the formula is true for certain definite positive integral value r of n. Then we must have

$$1^2+2^2+3^2+.....+r^2=\frac{r(r+1)(2r+1)}{6}.$$

Now add $(r+1)^2$ to both the sides.

$$\therefore \quad 1^2+2^2+3^2+\ldots\ldots+r^2+(r+1)^2$$

$$=\frac{r(r+1)(2r+1)}{6}+(r+1)^2$$

$$=\frac{r+1}{6}\{r(2r+1)+6(r+1)\}$$

$$=\frac{(r+1)(r+2)(2r+3)}{6}$$

which is exactly the same as would be got by giving to n the value $r+1$. So that we establish that if the formula is true for a certain integral number r, it is also true for the next integral number $r+1$.

But $1^2+2^2=5=\frac{2.3.5}{6}$.

It means that the formula is true when $n=2$. It must be, therefore true for $n=3$, and therefore for $n=4$, and so on.

Hence the formula is true for all *positive integral* values of n.

Notice that there are three steps of the process. *Firstly*, to show that if the formula is true for a certain number r, it is also true for $r+1$. *Secondly*, to show that the formula is true for the numbers 1, 2, 3. *Thirdly*, to generalise.

EXERCISE XXI

1. Sum $2^2+4^2+6^2+\ldots\ldots+$ to n terms.

2. Sum $1^2+3^2+5^2+\ldots\ldots$ to n terms.

3. Sum $1^3+3^3+5^3+\ldots\ldots$ to p terms.

4. Find the nth term and the sum of n terms of the series :—

(*i*) $1.2+2.3+3.4+\ldots\ldots$

(*ii*) $2.5+5.8+8.11+\ldots\ldots$

(*iii*) $3.5+5.7+7.9+\ldots\ldots$

(*iv*) $2.7+3.8+4.9+\ldots\ldots$

(*v*) $1.3.5+3.5.7+5\ 7.9+\ldots\ldots$

(*vi*) $3^2+7^2+11^2+\ldots\ldots$

(*vii*) $1.2^2+2.3^2+3.4^2+\ldots\ldots$

(*viii*) $1.3.6+2.4.8+3.5.10+\ldots\ldots$

5. Find the sum to n terms of the series whose general term is

(i) $(n+1)^2$ (ii) $(n+1)(n-2)$ (iii) $2n^2-1$.

(iv) $3n^3+2$ (v) 2^n+6n^2 (vi) $2n^3+3n^2+n$.

6. Show that $(2n-1)+2(2n-3)+3(2n-5)+$.....to n terms is equal to $1^2+2^2+3^2+$...to n terms.

7. Sum $1+(1+2)+(1+2+3)+$.....to n terms.

8. Sum $1^2+(1^2+2^2)+(1^2+2^2+3^2)+$.....to n terms.

9. Sum the squares of n terms of an A. P. whose first term is a and common difference d.

10. Find the nth term and hence the sum to n terms of :—

(i) $2+7+14+23+34+$.....

(ii) $1+3+6+10+15+$.....

(iii) $1+4+8+13+19+$.....

(iv) $3+13+29+51+79+$..... 19

(v) $1+5+11+19+29+$.....

11. Find the nth term and the sum to n terms of the series ;—

(i) $6+9+21+69+261+$.....

(ii) $5+7+11+19+$.....

(iii) $1+5+21+85+$.....

(iv) $1+3+7+15+31+$.....

12. Sum to n terms by finding the nth term of the series :—

(i) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$.....

(ii) $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\frac{1}{10.13}+$.....

(iii) $\frac{1}{p(p+2)}+\frac{1}{(p+2)(p+4)}+\frac{1}{(p+4)(p+6)}+$.....

13. Establish by Induction the formulæ :—

(i) $1+2+3+$.....$+n=\frac{1}{2}n(n+1)$.

(ii) $1^3+2^3+3^3+$.....$+n^3=\left[\frac{n(n+1)}{2}\right]^2$

(iii) $1.2+2.3+3.4+$.....$+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

(iv) $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$.....$+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$.

14. If $x<1$ and $y=1+4x+7x^2+10x^3+$.....*ad infinitum* ; show that $xy=1+y\pm\sqrt{1+3y}$.

Miscellaneous Questions II.

1. Sum up to n terms the series $0{\cdot}7+0{\cdot}71+0{\cdot}72+0{\cdot}73+\ldots\ldots\ldots\ldots$ (P. U. 1923)

2. There are n arithmetic means between 54 and 3 such that the $(n-7)$th mean is to the 3rd mean as 3 : 5; find n.

3. Four numbers in A. P. are such that the product of two extremes is 27 and the sum of the two means is 12; find them.

4. The sum of two numbers is n times their Geomr tric Mean. Compare the numbers.

5. Sum up to 9 terms the series $\frac{1}{2}\frac{0}{7}-\frac{3}{9}+\frac{4}{9}-\ldots\ldots\ldots\ldots$

6. Find the nth term of $4++4\frac{2}{7}+4\frac{8}{3}+5+\ldots\ldots\ldots\ldots$ (P. U. 1918)

7. In the A. P whose last term is 13 and common differ nce 2, the sum of how many terms will be 40 ?

8. Sum up to infinity the series $2+2+\frac{3}{2}+1+\frac{5}{8}+\ldots\ldots$

If S denotes the sum and Sn the sum of the first n terms of the series, find n for which $S-Sn<$ or $=\frac{1}{5}$. (P. U, 1933)

9. If n is odd, find the sum of $1+2^2+3+4^2+5+6^2+\ldots\ldots\ldots\ldots$to n terms.

10. If the mth term of a series in H. P. be n and the nth term be m, prove that the rth term is $\frac{mn}{r}$.

11. If $x=\frac{1}{5}$ find the value of $1+4x+9x^2+16x^3+25x^4+\ldots\ldots\ldots\ldots$to infinity.

12. Find the least value of n for which $S-Sn<{\cdot}03$, where S denotes the sum to infinity and Sn the sum to n terms of the series

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\ldots\ldots$ (P. U. 1932)

13. Sum the series :--

(*i*) $2+5+10+17+26+\ldots\ldots$to n terms. (P. U. 1921)

(*ii*) $1-x+2x^2-3x^3+4x^4-\ldots$to infinity. (P. U. 1932)

(*iii*) $1^2-2^2+3^2-4^2+\ldots\ldots\ldots+(2n-1)^2-(2n)^2$. (P. U. 1931)

14. Sum the series ;—

$n.1+(n-1).2+(n-2).3+(n-3).4+\ldots\ldots+1.n$.

15. If m is the A. M. of n consecutive integers, show that the sum of their cubes is $m\,n\,\{m^2+\frac{1}{4}(n^2-1)\}$. (P. U. 1927)

16. Oranges n in number are so placed on the ground that the distance between the first and second is one foot, between the second and the third three feet, between the third and fourth five feet, and so on. A child is asked to collect them one by one in a basket placed at the first orange. Find the distance walked over by the child after he has collected all.

17. What number must be subtracted from each of the numbers 13, 15, 19 that the remainders may be in H. P. ?

18. Divide 25 into 5 parts in A. P. such that the sum of the squares of the least and greatest is less by 1 than the sum of the squares of the other three.

19. Sum the series :—

(*i*) $1.3^2+2.4^2+3.5^2+$..... ...to n terms.

(*ii*) $1.2+2^2.3+3^2.4+$.....to n terms.

20. If a, b, c, be in A. P. and a, b, d, in H. P., show that $\frac{c}{d}=1-\frac{2(a-b)^2}{ab}$.

21. Show that two series in A. P. will have the same ninth term if their sums to n terms are in the ratio $3n+31 : 5n-3$.

22. Sum to n terms the series $1^3-3^3+5^3-7^3+$.....

23. If x be the G. M. and a, b two Arithmetic Means between two numbers ; show that $x^2=(2a-b)(2b-a)$.

24. If $a^2+b^2-c^2$, $b^2+c^2-a^2$, $c^2+a^2-b^2$ be in G. P., show that $\frac{a^2}{c^2}+\frac{c^2}{b^2}$, $\frac{b^2}{c^2}+\frac{c^2}{b^2}$, $\frac{a^2}{b^2}+\frac{b^2}{c^2}$ are in A. P.

25. If $a_1, a_2, a_3,$.....a_n be in H. P., show that

$$a_1a_2+a_2a_3+a_3a_4+\text{.....}+a_{n-1}a_n=(n-1)a_1a_n.$$

[Let d be the common difference of the corresponding A. P. Then

$$d=\frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{a_3}-\frac{1}{a_2}=\text{.....}=\frac{1}{a_n}-\frac{1}{a_{n-1}}$$

$$=\frac{\frac{1}{a_n}-\frac{1}{a_1}}{n-1};$$

and $d = \frac{a_1 - a_2}{a_1 a_2} = \frac{a_2 - a}{a_2 a} = \ldots = \frac{a_{n-1} - a_n}{a_{n-1} a_n}$

$$= \frac{a_1 - a_n}{a_1 a_2 + a_2 a_3 + \ldots\ldots + a_{n-1} a_n}$$ Hence.....]

26. If in the equation

$$(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 = 0$$

the quantities a, b, c, d and x are all real, show that a, b, c, d form a G. P. with common ratio x.

[The given equation is

$$(ax - b)^2 + (bx - c)^2 + (cx - d)^2 = 0 \quad \therefore \quad ax - b = bx - c$$
$$= cx - d = 0, \text{ and so on.}]$$

27. If Sr denotes the sum of the rth powers of the terms of an infinite G P. whose first term is a and common ratio x, then show that

$$\left(\frac{1}{S_1} + \frac{1}{S_3} + \frac{1}{S_5} + \ldots\ldots\right)^2$$
$$-\left(\frac{1}{S_2} + \frac{1}{S_4} + \frac{1}{S_6} + \ldots\ldots\right)^2$$
$$\ldots \frac{a^2}{a^2 - 1} \cdot \frac{(1 - x)^2}{a^2 - x^2} \cdot$$

[$Sr = a^r + a^r x^r + a^{2r} x^{2r} + \ldots\ldots = \frac{a^r}{1 - x^r}$.

$\therefore \quad S_1 = \frac{a}{1 - x}$. $S_2 = \frac{a^2}{1 - x^2}$ and so on.

Substitute and simplify.]

28. Find the sum of all fractions of the form $\frac{a^p}{b^q}$, where p and q have all positive integral values from zero to $n - 1$.

29. In an A. P., a is the first term and d the common difference ; show that the sum of the cubes of n terms is

$$na^3 + \tfrac{3}{2}n(n-1)a^2 d + \tfrac{1}{4}n(n-1)(2n-1)ad^2 + \tfrac{1}{4}n^2(n-1)^2 d^3.$$

30. Between a and b are inserted n Airthmetic and n Harmonic Means, and a series of n terms is formed by dividing each A. M. by the corresponding H. M, show that the sum of the series is

$$n\left(1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right).$$

(If A_r and H_r be the rth A. M. and H. M. respectively

then $A_r=\frac{a(n+1)+r(b-a)}{n+1}$ and $H_r=\frac{(n+1)ab}{b(n+1)+r(a-b)}$.

Then prove that $\frac{A_r}{H_r}=1+\frac{r(a-b)^2}{(n+1)ab}-\frac{r^2(a-b)^2}{(n+1)^2ab}$.

Now give to r values 1 to n and add

$\therefore$ Sum $=n+\frac{(a-b)^2}{(n+1)ab}\{1+2+3+\ldots\ldots+x\}$

$-\frac{(a-b)^2}{(n+1)^2ab}\{1^2+2^2+3^2+\ldots\ldots+n^2\}$ etc., etc.).

31. Sum $\frac{1}{(1+n)(2+n)}+\frac{1}{(2+n)(3+n)}+\frac{1}{(3+n)(4+n)}$
$+\ldots\ldots\ldots$to n terms.

32. Sum $\frac{1}{(1+n)(1+n^2)}+\frac{n}{(1+n^2)(1+n^3)}$
$+\frac{n^2}{(1+n^3)(1+n^4)}+\ldots\ldots\ldots$to n terms.

33. Sum $\frac{2+n}{(1+n)(3+2n)}+\frac{2+2n}{(3+2n)(5+4n)}$
$+\frac{2+4n}{(5+4n)(7+8n)}+\ldots\ldots\ldots$*ad infinitum.*

34. Sum the infinite series

$$\frac{x}{1-x^2}+\frac{x^2}{1-x^4}+\frac{x^4}{1-x^8}+\ldots\ldots\ldots$$

($\frac{x}{1-x^2}$ is the sum of the G. P. whose first term is x and common ratio x^2;

$\therefore$ $\frac{x}{1-x^2}=x+x^3+x^5+\ldots$..Similarly $\frac{x^2}{1-x^4}=x^2+x^6$
$+x^{10}+\ldots\ldots$; and so on.

Hence the sum $=x+x^2+x^3+x^4\ldots\ldots$).

CHAPTER IX

Permutations and Combinations.

51. Important Principle. Problems of the following type often arise in everyday life :—

Ex. 1. In how many ways can two students select a book each from a set of 4 books ?

Obviously the first student has 4 ways of selection at his disposal. After the first student has made his selection *which he can do in 4 ways*, the second can select a book in three ways. Thus if a, b, c, d be the names of the books, then the different selections can be tabulated as below :—

1st student	a	b	c	d
2nd student	b, c, d	a, c, d	a, b, d	a, b, c

This table shows, as is otherwise also clear, that with each selection by the first student are associated three different selections by the second student. The number of the first selections being four, it follows that the total number of selections by the two students is $4 \times 3 = 12$.

The above example is a particular case of the **general theorem :—**

If one operation can be performed in m *different ways, and when it has been performed in any one of these ways, a second operation can then be performed in* n *different ways ; the number of ways of performing the two operations together is* $m \times n$.

If the first operation can be performed in *one* way only and the second in n ways, the total number of ways of performing the two operations is n.

If the first operation can be performed in *two* ways and the second in n ways, the total number of ways of performing the two operations is $2 \times n$

If the first operation can be performed in *three* ways and the second in n ways. the total number of ways of

performing the two operations is $3 \times n$.

So on and so forth.

Hence generalising, if the first operation can be performed in m ways and the second in n ways, the total number of ways of performing the two operations is $m \times n$.

Cor. If one operation can be performed in m ways, a second operation in n ways, a third in p ways, and so on ; then the number of ways of performing all these operations is $m \times n \times p \times \ldots\ldots$

Ex. 2. In how many ways can a class time-table of the first four periods be framed when the number of subjects taught to that class is 9 ?

The first period can be filled in by 9 different subjects.

When it has been done, the second period can be filled in 8 different ways ; and with each of these ways are associated the 9 ways of filling the first period. Thus the number of ways of filling the first two periods is 9×8.

After the first two periods, the third can be filled in 7 different ways since there are only 7 subjects left ; and with each of these ways are associated the 9×8 ways of filling the first two periods. Thus the number of ways of filling the first three periods is $9 \times 8 \times 7$.

Lastly, as there are 6 subjects left, the fourth period can be filled in 6 different ways ; and with each of these ways are associated the $9 \times 8 \times 7$ ways of filling the first three periods. Hence the total number of ways of filling the first four periods is $9 \times 8 \times 7 \times 6 = 3024$.

52. Definition. In the Example 2 of Art. 51, all possible *arrangements* of a number of things taken some at a time have been obtained.

Each of the *arrangements* which can be made by taking some or all of a number of things is called a **Permutation**.

The number of permutations of n different things taken r of them at a time is symbollically denoted by ${}^{n}P_{r}$.

53. To find the number of permutations of n different things taking *r* of them at a time.

In other words *to find the value of* ${}^{n}P_{r}$.

In the light of the Example 2 of Art. 51 this proposition means to find the number of ways of filling *r* places out of the *n* given things.

To fill the first place there are *n* things at our disposal. Thus there are *n* ways of filling the first place.

To fill the second place, $n-1$ things are left with us. The second place can, therefore, be filled in $n-1$ ways. But with each of these $n-1$ ways are associated the *n* ways of filling the first place. Thus the first two places can be filled in $n(n-1)$ ways.

To fill the third place we have $n-2$ things left. The third place can, therefore, be filled in $n-2$ ways. But with each of these ways are to be considered the $n(n-1)$ ways of filling the first two places. Thus the first three places can be filled in $n(n-1)(n-2)$ ways.

Proceeding in this way we see that the number of places filled up is the same as the number of factors obtained.

Thus the number of ways of filling the *r* places

$= n(n-1)(n-2)\ldots\ldots$to *r* factors

$= n(n-1)(n-2)\ldots\ldots(n-r-1)$

$= n(n-1)(n-2)\ldots\ldots(n-r+1)$.

Hence we find that ${}^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1)$.

Cor. 1. The number of permutations of *n* things taking all at a time is given by

${}^{n}P_{n} = n(n-1)(n-2)\ldots\ldots$to *n* factors

$= n(n-1)(n-2)\ldots\ldots 3.2.1$.

Now the product of all natural numbers upto *n* is usually denoted by the symbol $\underline{n}$ or $n!$ which is read "factorial *n*".

Thus ${}^{n}P_{n} =$ $\underline{n}$.

Cor. 2. ${}^{n}P_{r} = n(n-1)(n-2)\ldots\ldots(n-r+1)$

$= \dfrac{n(n-1)\ldots(n-r+1)(n-r)\ldots 3.2.1}{(n-r)\ldots\ldots 3.2.1}$

$$= \frac{\text{product of all natural numbers upto } n}{\text{product of all natural numbers upto } n-r}$$

$$= \frac{n}{n-r}.$$

54. A second proof of the above proposition may also be given as below :—

To find the value of ${}^nP_r$ let us denote the n different things by a, b, c, d.....

If we ignore any one of the things, a (say), the number of permutations of the remaining $n-1$ things taking $r-1$ at a time will be ${}^{n-1}P_{r-1}$. Now with each of these permutations combine the thing a. Thus the number of such permutations of n things taken r at a time as begin with a is ${}^{n-1}P_{r-1}$. Similarly all such permutations as begin with b is ${}^{n-1}P_{r-1}$ And so for all the n things.

$$\therefore \quad {}^nP_r = {}^{n-1}P_{r-1} + {}^{n-1}P_{r-1} + {}^{n-1}P_{r-1} + \text{...to } n \text{ terms}$$
$$= n.\ {}^{n-1}P_{r-1}$$

By a similar reasoning

$${}^{n-1}P_{r-1} = (n-1).{}^{n-2}P_{r-2}$$

and ${}^{n-2}P_{r-2} = (n-2).{}^{n-3}P_{r-3}$

.....

$${}^{n-r+2}P_2 = (n-r+2).{}^{n-r+1}P_1$$

$${}^{n-r+1}P_1 = n-r+1.$$

Multiply together the corresponding sides of all these equalities and cancel like factors from each side, then

$${}^nP_r = n(n-1)(n-2).....(n-r+1).$$

Ex. 1. How many numbers between 99 and 1000 can be formed with the digits 2, 3, 7, 0, 8, 6 ?

The numbers between 99 and 1000 are made up of 3 digits. Out of the given 6 digits we have to take 3 at a time. Thus the total number of such numbers is ${}^6P_3$.

But from these we have to reject all such numbers as have zero on the left. Now the place of zero being fixed only 2 digits out of 5 are required to form such numbers ; their number therefore is ${}^5P_2$.

Hence the required number of numbers

$$= {}^6P_3 - {}^5P_2 = 6.5.4 - 5.4 = 210 - 20 = 190.$$

Ex. 2. How many numbers formed by taking all the digits 1, 2, 3, 4, 5 are (*i*) even, (*ii*) less than 40,000.

(*i*) An even number must end in 2 or 4. The place of 2 being fixed we have to arrange the remaining 4 digits. Thus the number of numbers ending in 2 is ${}^4P_4$. And the same is the number of numbers ending in 4.

∴ the number of even numbers $= {}^4P_4 + {}^4P_4 = 4 + 4$
$= 48.$

(*ii*) The numbers less than 40,000 must have 1, 2, or 3 on the left. The place of 1 being fixed, the remaining can be permuted in ${}^4P_4$ ways. The same is true for 2 and 3.

∴ the number of numbers less than $40{,}000 = 3.\ {}^4P_4$
$= 3 \quad 4 = 72.$

Ex. 3. There are 6 flags of different colours. How many signals can be given with them when any number of flags may be used at a time ?

The number of signals when one flag may be used $= {}^6P_1 = 6$
,, ,, ,, ,, ,, 2 flags ,, ,, $= {}^6P_2 = 30$
,, ,, ,, ,, ,, 3 flags ,, ,, $= {}^6P_3 = 120$
,, ,, ,, ,, ,, 4 flags ,, ,, $= {}^6P_4 = 360$
,, ,, ,, ,, ,, 5 flags ,, ,, $= {}^6P_5 = 720$
,, ,, ,, ,, ,, 6 flags ,, ,, $= {}^6P_6 = 720$

∴ the total number of signals $= 1956.$

Ex. 4. How many words can be made with the letters of the word *auction*, the vowels being always together ?

Considering all the four vowels as one, the number of letters is 4. Thus the number of arrangements is ${}^4P_4$.

But the four vowels can be arranged among themselves in ${}^4P_4$ ways.

Hence the required number of words $= {}^4P_4 \times {}^4P_4$
$= 24 \times 24 = 576.$

Ex. 5. In how many words framed by the different arrangements of the letters of the word *strange*, the letters *s*, *t*, *r* will never come together ?

Suppose that the letters *s*, *t*, *r* come always together. Then we have to arrange the five letters (*str*), *a*, *n*, *g*, *e*. This can be done in ${}^5P_5$ ways. But in each of these arrangements the three letters *s*, *t*, *r* can be arranged among themselves in ${}^3P_3$ ways. Hence the total number of arrangements in which *s*, *t*, *r* are never separated, is ${}^5P_5 \times {}^3P_3 = \underline{5} \times \underline{3} = 720.$

But the total number of arrangements of all the seven letters is ${}^7P_7 = 7 = 5040$.

Hence the number of arrangements in which s, t, r never come together $= 5040 - 720 = 4320$.

Ex. 6. In how many ways can n persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements ?

To seat at a round table the place of one man shall have to be fixed. The remainig $n-1$ men can be arranged in $\lfloor n-1$ ways. But this number includes all those arrangements in which the neighbours are exchanged. The number of neighbours in each case being 2, we have to count such a pair of arrangements as only one. Hence the required member of ways is $\frac{1}{2} \lfloor n-1$.

EXERCISE XXII.

1. Find the value of :—

(*i*) ${}^5P_3$, ${}^{10}P_2$, ${}^8P_6$, ${}^{18}P_1$.

(*ii*) $\lfloor 7$, $\frac{\lfloor 12}{\lfloor 9}$, $\frac{\lfloor 14}{\lfloor 16}$, $\frac{\lfloor 9 \times \lfloor 6}{\lfloor 16}$, $\frac{\lfloor 15}{8 \times \lfloor 7}$,

$\frac{7 \times \lfloor 8}{10 \times \lfloor 5}$, $\frac{(8!)^2}{(4!)^4}$.

2. Prove that :—

(*i*) $15.14. \lfloor 13 = 15$. (*ii*) $n \quad n-1 = n$

(*iii*) $\lfloor n = n(n-1)\ldots(n-r+1) \lfloor n-r$.

(*iv*) ${}^nP_r = \frac{n}{\lfloor n-r}$. (*v*) ${}^nP_n = n \times {}^{n-1}P_{n-1}$.

(*vi*) ${}^nP_r = n.{}^{n-1}P_r$

3. Find the value of n if :—

(*i*) ${}^nP_2 = 30$. (*ii*) ${}^nP_3 = 30 \times {}^nP_2$.

(*iii*) ${}^nP_4 = \frac{7}{8}{}^{n+1}P_4$. (*iv*) ${}^nP_5 = 20\ {}^nP_4$.

(*v*) ${}^{2n}P_{n-1} = \frac{1}{11}\, 2^{n-1}P_n$.

(*vi*) ${}^9P_n = 2 \times {}^9P_{n-1}$.

4. Show that (*i*) $2n = 2^n\{1.3.5\ldots(2n-1)\} \quad n$.

(*ii*) ${}^{2n}P_n = 2^n\{1.3.5\ldots(2n-1)\}$.

5. A class room has 5 doors. In how many ways can a student come in and go out by different doors ?

6. There are 5 seats on a bench. In how many different ways can 3 students sit on it ?

7. On a shelf there are 10 books. In how many ways can 10 students make their selections of one book each from the shelf

8. How many words can be framed by the different arrangements of all the letters of the word *circut* ?

9. How many arrangements of the letters of the word *camphor* will begin with *c* !

10. How many different numbers can be formed with the digits 1, 4, 5, 6, 3, 2 ? How many of them will be even ?

11. In how many ways can the digits 3, 2, 7, 4, 6, 5 be arranged to form numbers between 999 and 10,000 ? How many of them will begin with 2 and end with 3 ?

12. How many signals can be given with 7 different flags using only 4 at a time ?

13. How many signals can be given with 7 different flags when not more than 4 are used at a time ?

14. How many signals can be given with 7 different flags when not less than 4 are used at a time ?

15. Six students enter a railway compartment fitted with ten seats. In how many ways can they be seated ?

16. How many permutations of the letters of the word *history* will begin with *h* and end in *y* ?

17. In the permutations of the letters of the word *history* how many will have *s*, *t*, *r* (*i*) always together : (*ii*) never together ?

18. In how many permutations of the letters of the word *history* the letters, *i*, *o* will occupy only the odd places ?

19. In how many of the arrangements of *r* books on a shelf two particular books will never be together ?

20. How many words can be formed out of the letters of the word *question* and how many of them will

(*i*) begin with *q* ?

(*ii*) begin with *q* and end in *n* ?

(*iii*) have *st* always together ?

(*iv*) have *sti* always together ?

(*v*) have *sti* never together ?

(*vi*) have a vowel in the beginning and end ?

21. In how many ways can 6 students form a ring ?

22. In how many ways can 9 boys stand in a ring so that none will have the same neighbours in any two arrangements ?

23. In how many ways can 6 Americans and 6 Englishmen sit at a round table so that no two Amercians sit together ?

24. How many arrangements of the letters of the word *brave* taken four at a time will contain *b* and how many will begin with *b* ?

25. How many signals can be given with 7 different flags when any number of them may be hoisted at a time ?

26. In how many ways can 4 prizes be given away to 7 students when a student may get any number of prizes ?

27. There are 4 books on History, Mathematics, and Economics each and 3 books on Philosophy and Physics each ; in how many ways can these 18 books be arranged on a shelf so that books on the same subject are not separated ?

28. In a cycle race 7 competitors are to start. Find the number of ways in which they can initially stand so that (*i*) a particular competitor may always have the same competitor on his left, (*ii*) that a particular competitor may always be between the same two competitors.

Combinations.

55. Yet another type of problems that often arise will be illustrated by the following example :—

Ex. An examination paper contains four questions and only three are to be answered ; in how many ways can a candidate make his selection ? In how many ways can he write the answers ?

Notice that here we have nothing to do with the arrangements of the questions or answers. We have only to consider the different selections that can be made.

If the questions are marked I, II, III, IV, evidently the different selection will be (1) I, II. III ; (2) I, II, IV ; (3) I, III, IV : and (4) II. III, IV ;

Thus in 4 ways can a candidate make his selection of 3 from 4 questions.

Again, if in addition to the number of *selections* we have to consider the number of *arrangements* in which a candidate can write his answeres, we notice that each selection gives rise to 6 different arrangements ; *viz.*

(1) I, II, III ; I, III, II ; II, III, I,
III, II, I : II, I, III, ; III, I, II ;
(2) I, II, IV ; I, IV, II ; II, IV, I ;
IV, II, I ; II, I, IV ; IV, I, II ;
(3) I, III, IV : I, IV, III ; III, IV, I ;
IV, III, I ; III, I, IV ; IV, I, III :
(4) II, III, IV ; II, IV, III ; III, IV, II ;
IV, III, II ; III, II, IV ; IV, II, III.

Hence the number of selections of 4 things taken 3 at time is 4, whereas the number of arrangements of 4 things taken 3 at a time is 24.

Note The difference between *selection* and *arrangement* should be carefully noted. In finding arrangements we have to consider the order in which the things are placed ; and in finding the selections we do not consider the order but we pay attention only to the number of things that are contained in each selection.

56. Definition. *Each selection or group which can be formed by taking some or all of a number of things, is called a* **combination.**

The synonyms for permutation are arangement implying order and those for combination are selection or group.

The number of combinations of n things taken r at a time is usually denoted by ${}^{n}C_{r}$.

Notice that ${}^{n}P_{r}$ and ${}^{n}C_{r}$ are mere positive integral numbers and cannot, therefore be fractional or negative.

57. To find the number of combinations of n different things taken r at a time.

In other words *to find the value of* ${}^{n}C_{r}$.

Suppose that x is the required number of combinations.

Then $x = {}^{n}C_{r}$.

Now each of these combinations contains r different things which among themselves can be arranged in ${}^{r}P_{r}$ ways

i. e., in $\lfloor r$ ways.

Thus one combination gives rise to r permutations.

$\therefore$ x combinations will give $x \times r$ permutations.

But the total number of permutations of n things taken r at a time is ${}^{n}P_{r}$.

$\therefore \quad x \times \lfloor r = {}^{n}P_{r}$.

$$\therefore \quad x = \frac{{}^{n}P_{r}}{\lfloor r} = \frac{\lfloor n}{\lfloor r \; \lfloor n-r}$$

Hence $$ {}^{n}C_{r} = x = \frac{\lfloor n}{\lfloor r \; \lfloor n-r}.$$

Cor. *To find the value of* $\lfloor 0$.

In the above formula, put $n = r$, then

$${}^{n}C_{r} = \frac{\lfloor n}{\lfloor n \; \lfloor n-n} = \frac{1}{\lfloor 0}$$

But ${}^{n}C_{n}$ denotes the number of combination of n things taken all at a time; and such number is only 1.

$\therefore \quad {}^{n}C_{n} = 1$

$\therefore \quad 1 = \frac{1}{\lfloor 0}$

$\therefore \quad \lfloor 0 = 1.$

Thus $\lfloor 0$, though meaningless, stands for unity.

58. A second proof of the above proposition may also be given.

Denote the number of combinations of n things taken r at a time by ${}^{n}C_{r}$ and suppose the different things are a, b, c, d,.....

From the number ${}^{n}C_{r}$ to find such combinations that contain a, we have only to find the number of combinations of $n-1$ letters b, c, d,.....taken $r-1$ at a time Thus there are ${}^{n-1}C_{r-1}$ combinations which contain the letter a, *i.e.*, the letter a is written ${}^{n-1}C_{r-1}$ times in all the combina-

tions. Similarly every other letter is written ${}^{n-1}C_{r-1}$ times in all the combinations. Thus there being n letters in all, the total number of letters written in all the combinations is $n \times {}^{n-1}C_{r-1}$.

Again each of the ${}^{n}C_{r}$ combinations contains r letters; so that the total number of letters written in all the combinations is $r \times {}^{n}C_{r}$

Hence $r \quad {}^{n}C_{r} = n \times {}^{n-1}C_{r-1}$

or $${}^{n}C_{r} = \frac{n}{r} \times {}^{n-1}C_{r-1}$$

By a similar reasoning, we have

$${}^{n-1}C_{r-1} = \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}$$

$${}^{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}^{n-3}C_{r-3}$$

. . . . . .

$${}^{n-r+2}C_{2} = \frac{n-r+2}{2} \times {}^{n-r+1}C_{1}$$

$${}^{n-r+1}C_{1} = n-r+1.$$

Multiply together the vertical columns and cancel like factors from both sides. Thus we have

$${}^{n}C_{r} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{(n-2)\ldots(n-r+1)}{(r-2)\ldots3.2.1}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r}.$$

Notice that the value of ${}^{n}C_{r}$ derived here is the same as got in the last article. For

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r} = \frac{n(n-1)\ldots(n-r+1)}{\underline{r} \quad \underline{n-r}} \cdot \frac{n-r}{}$$

$$= \frac{n}{r \quad n-r}.$$

Note. This proof has been derived independently of the formula for ${}^{n}P_{r}$.

59. *To prove that the number of combinations of* n *things taken* r *at a time is the same as the number of combinations of* n *things txken* n−r *at a time.*

In other words *to prove that* ${}^{n}C_{r} = {}^{n}C_{n-r}$.

From n things when we form a group of r things we leave out a group of $n-r$ things. It means that for every group of n things there corresponds a group of $n-r$ things. Thus the number of groups containing r things each is the same as the number of groups containing $n-r$ things each.

Hence ${}^{n}C_{r}={}^{n}C_{n-r}$.

Or thus: Since ${}^{n}C_{r}=\frac{\lfloor n}{\lfloor r \,\lfloor n-r}$

$$\therefore \quad {}^{n}C_{n-r}=\frac{\lfloor n}{\lfloor n-r \,\lfloor n-(n-r)}$$

$$=\frac{\lfloor n}{\lfloor n-r \,\lfloor r}.$$

Hence ${}^{n}C_{r}={}^{n}C_{n-r}$.

Note. Such combinations are called *complementary combinations.*

Cor. *To find the value of* ${}^{n}C_{0}$.

In the formula ${}^{n}C_{n-r}=\frac{\lfloor n}{\lfloor r \,\lfloor n-r}$ put $n=r$.

Then ${}^{n}C_{0}=\frac{\lfloor n}{\lfloor n \,\lfloor n-n}=\frac{1}{\lfloor 0}=1$.

60. *To prove that* ${}^{n}C_{r-1}+{}^{n}C_{r}={}^{n+1}C_{r}$.

The number ${}^{n+1}C_{r}$, *i.e.*, the number of combinations of $n+1$ things taken r at a time, can be divided into two parts :—

(*i*) Such combinations that contain a particclar thing, and

(*ii*) Such combinitions that do not contain that particular thing.

Now the number of combinations in the first part can be obtained by selecting $r-1$ things from the remaining n things. The number of such combinations is ${}^{n}C_{r-1}$.

And the number of combinations in the second part can be obtained by selecting r things from the remaining n things. The number of such combinations is ${}^{n}C_{r}$.

Hence ${}^{n}C_{r-1}+{}^{n}C_{r}={}^{n+1}C_{r}$.

Or thus : From the definition

$$
\begin{aligned}
{}^{n}C_{r-1}+{}^{n}C_{r} &= \frac{n}{r-1}\,\frac{}{n-r+1}+\frac{}{r}\,\frac{n}{n-r} \\
&= \frac{n}{r-1}\,\frac{(n-r+1)}{n-r}+\frac{r}{r}\,\frac{n}{r-1}\,\frac{}{n-r} \\
&= \frac{n}{r-1}\,\frac{}{n-r}\left\{\frac{1}{n-r+1}+\frac{1}{r}\right\} \\
&= \frac{n}{r-1}\,\frac{}{n-r}\times\frac{n+1}{r(n-r+1)} \\
&= \frac{n+1}{r}\,\frac{}{n-r+1} \\
&= {}^{n+1}C_{r}.
\end{aligned}
$$

Ex. 1. How many triangles can be formed by joining the angular points of a decagon ? (P. U. 1891)

Of 10 angular points of a decagon, any 3 will give a triangle.

$\therefore$ The number of triangles $= {}^{10}C_{3} = \frac{10\times 9\times 8}{3\times 2\times 1} = 120.$

Ex. 2. There are n points in a plane, no three of which are in the same straight line, with the exception of p points which are all in the same straight line : find the number of straight lines which can be formed by joining them.

First consider that of all the n points no three are in the same straight line. Then since any 2 of them will give a straight line, the number of straight lines will be ${}^{n}C_{2}$.

From this we should subtract ${}^{p}C_{2}$ which is the number of straight lines got by joining any 2 of the p points, since these p points do not give different straight lines.

And we should add 1 to the number, since p points give only one straight line.

Hence the total number of lines

$$= {}^{n}C_{2}-{}^{p}C_{2}+1 = \frac{n(n-1)}{2}-\frac{p(p-1)}{2}+1.$$

Ex. 3. From 8 Indians and 4 Burmese a committee of 5 is to be formed ; in how many ways can this be done when the committee contains (*i*) exactly 2 Burmese, (*ii*) at least 2 Burmese ?

(*i*) We have to select 3 Indians out of 8, and 2 Burmese out of 4. The two selections can be made in ${}^8C_3$ and ${}^4C_2$ ways respectively.

Since each of the first groups can be associated with each of the second, the required number of ways

$$={}^8C_3 \times {}^4C_2 = 336.$$

(*ii*) The committee may contain 2, 3 or 4 Burmese, and therefore respectively 3, 2 or 1 Indian.

Hence the required number of ways

$$={}^4C_2 \times {}^8C_3 + {}^4C_3 \times {}^8C_2 + {}^4C_4 \times {}^8C_1$$
$$=6 \times 56 + 4 \times 28 + 1 \times 8$$
$$=336 + 112 + 8$$
$$=456.$$

Ex. 4. At an election, where every voter may vote for any number of candidates not greater than the number to be voted, there are 6 candidates and three members are to be chosen ; in how many ways a man may vote ?

A man can vote for 1, 2 or 3 candidates out of 6.

Hence the required number of ways

$$={}^6C_1 + {}^6C_2 + {}^6C_3$$
$$=6 + 15 + 20$$
$$=41.$$

Ex. 5. In how many ways can 8 books on History and 6 on Economics be put on a shelf so that no two books on Economics may be together ?

Let the History books, represented by H's, be placed on the shelf leaving space on either side of it for a book on Economics.

$$\times H \times H \times H \times H \times H \times H \times H \times H \times$$

Clearly there are (8+1) places for 6 books on Economics.

Hence the required number of ways $={}^9C_6 = 84$.

Ex. 6. Out of 17 consonants and 5 vowels how many words can be formed. each consisting of 2 vowels and 3 consonants. (P. U. 1893)

From 5 vowels 2 can be selected in ${}^5C_2$ ways; and from 17 consonants 3 can be selected in ${}^{17}C_3$ ways.

Thus the total number of group of 5 letters each

$$= {}^5C_2 \times {}^{17}C_3.$$

But in order to form different words, 5 letters of each group can be arranged in ${}^5P_5$ ways.

Hence the total member of words formed

$$= {}^5C_2 \times {}^{17}C_3 \times {}^5P_5$$
$$= 10 \times 680 \times \underline{5}$$
$$= 816,000.$$

Ex. 7. Find the member of ways in which $p+q$ things can be divided into two groups containing p and q things.

Put $p+q=n$ For every group of p things there corresponds a group of q things left behind.

$\therefore$ the required number of ways $= {}^nC_p$

$$= \frac{n}{p \quad n-p} = \frac{p+q}{p \quad q}$$

Note :-It can be similarly shown that $p+q+r$ things can be divided into three groups containing p, q and r things in

$$\frac{p+q+r}{p \quad q \quad r} \text{ ways.}$$

Ex. 8 Given n, find r for which ${}^nC_r$ is greatest.

We know that

$${}^nC_r \quad \frac{n(n-1)(n-2).....(n-r+2)(n-r+1)}{1.\ 2.\ 3.... .....(r-1)r.}$$

and that ${}^nC_{r-1} = \frac{n(n-1)(n-2).....(n-r+2)}{1.\ 2.\ 3.....(r-1)}$

$$\therefore \quad {}^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}.$$

Thus we see that the value of ${}^nC_r$ depends upon the values of ${}^nC_{r-1}$ and $\frac{n-r+1}{r}$.

The factor $\frac{n-r+1}{r}$ or $\frac{n+1}{r}-1$ decreases in value as r increases. Thus as r takes the values 1, 2, 3.....in

succession, ${}^{n}C_r$ increases in value till $\frac{n-r+1}{r}$ is equal to or less than 1.

$\therefore$ ${}^{n}C_r$ is greater than ${}^{n}C_{r-1}$

so long as $\frac{n-r+1}{r} > 1$

or $n+1 > 2r$

or $r < \frac{n+1}{2}$

Now (1) if n is *even*, $n+1$ is odd and, therefore, the greatest value of r is $\frac{n}{2}$.

Hence ${}^{n}C_r$ is greatest when $r = \frac{n}{2}$.

And (2) if n is *odd*, $n+1$ is even and, therefore, the greatest value of r is $\frac{n+1}{2}$. In this case ${}^{n}C_r = {}^{n}C_{r-1}$, *i.e.*, ${}^{n}C_{\frac{n+1}{2}} = {}^{n}C_{\frac{n-1}{2}}$; and each greater than any other value of ${}^{n}C_r$.

EXERCISE XXIII.

1. Find the values of ${}^{6}C_3$, ${}^{15}C_3$, ${}^{18}C_{15}$, ${}^{48}C_{47}$, ${}^{2r}C_r$.

2. Solve the following equations :—

(*i*) ${}^{x}C_7 = {}^{x}C_6$. (*ii*) $3.{}^{x}C_4 = 5.{}^{x}C_2$.

(*iii*) ${}^{9}C_3 = {}^{9}C_x$. (*iv*) $8 \times {}^{x}C_4 = {}^{x+1}C_5$.

(*v*) ${}^{x}P_6 = 360 \times {}^{x}C_6$. (*vi*) ${}^{x}P_4 = 30 \times {}^{x}C_5$.

3. If ${}^{n}C_{10} = {}^{n}C_{11}$, find ${}^{n}C_{15}$ and ${}^{n}C_5$.

4. Find n and r when ${}^{n}C_{r-1} : {}^{n}C_r : {}^{n}C_{r+1} = 2 : 3 : 4$.

5. If $r = {}^{m}C_2$, show that ${}^{r}C_2 = 3 \times {}^{m+1}C_4$.

6. Prove that ${}^{10}C_6 = {}^{9}C_6 + {}^{9}C_5$.

7. If the number of combinations of $2r+1$ things taken $2r-1$ at a time be 78, find the number of permutations of r things taken $r-4$ at a time.

8. Find the number of triangles formed by joining the angular points of a polygon of n sides.

9. How many diagonals can a polygon of n sides have ?

10. There are 15 points in a plane, no three of which are in the same straight line ;

(*i*) how many striaght lines can be drawn between them ?

(*ii*) how many triangles can be formed with them ?

(*iii*) if a fifteen-sided polygon be formed by joining them, how many diagonals with it have ?

11. A new railway line is constructed with 28 stations on it. How many different single journey tickets should be prepared for it ?

12. How many different committees of 7 members consisting of 4 Hindus and 3 Muslims can be made from 7 Hindus and 6 Muslims ?

13. In how many ways can a party of 4 be selected to play tennis from 9 players (*i*) when a special player is always to play ; (*ii*) when a special player is never to play ?

14 There are 16 players of cricket. In how many ways can a team of 11 be chosen from them (*i*) when 2 particular players are always included, (*ii*) when 2 particular players are always excluded ?

15. There are 18 points in a plane, no three of which are in the same straight line, with the exception of 5 points which are all in the same straight line. Find by joining them (*i*) how many different straight lines can be formed ; (*ii*) how many different triangles can be formed ?

16. In how many combinations of n things taken r at a time will two given things occur ?

17. From 10 persons a party of 4 is to be selected. In how many of the selections a particular person will always be (*i*) included, (*ii*) excluded ? (*iii*) Hence show that ${}^{10}C_4 = {}^9C_3 + {}^9C_4$.

18. Out of 7 consonants and 6 vowels how many words can be formed each consisting of 4 consonants and 2 vowels ?

19. From 4 consonants. 4 vowels and 4 capitals how many words can be made each consisting of 3 consonants, 2 vowels and beginning with a capital ?

20. A father takes his 8 children 3 at a time to cinema so often as he can form different groups. How often will he go and how often will each child go ?

21. In how many ways can 8 red balls and 6 black balls be arranged in a row so that no two black balls may be together ?

22. In the combinations of $4x$ things taken x together, show that the ratio of the number of combinations in which a particular thing occurs to the number of combinations in which it does not occur is 1 : 3.

23. In a class 90 students take admission. In how many ways can they be divided into two sections of 50 and 40 students ?

24. In how many ways can 21 men be divided into 3 parties of 7 men each ?

25. In how maay ways can 12 persons be seated for dinner on three tables when there are 3 seats on the first table, 4 on the second and 5 on the third, arrangements not to be concerned ?

61. So far we have considered permutations and combinations of things which were all *dissimilar.* The case of things some of which are similar is also worthy of consideration.

By the words *dissimilar, different* and *unlike* we mean things apparently distinguishable. For instance, in the word *similar* there being 3 vowels in all, the different vowels are only two ; so that the number of permutations got by the different arrangements of the letters will not all be different. The interchange of two *i's* in any arrangement will not give two different permutations. If, however one of the two *i's* be replaced by some other vowel say *a* the permutation got by the interchange of *i* and *a* will be quite different.

62. *To find the number of permutations of* n *things taken all together when* p *of the things are alike and the rest all different.*

Let x be the required number of permutations.

If the p things be arranged among themselves we shall get $\lfloor \underline{p}$ permutations, each of which should be associated with x permutations in order to get the total

number of permutations of n things taken all together, which number is n.

Hence we have $x \times p = n$

$$i.e.\ x = \frac{n}{p}.$$

Cor. The number of permutations of n things taken all together when p of them are alike of one kind, q of them alike of a second kind, r of them alike of a third kind, and so on and the rest all different, is

$$\frac{n}{p\ q\ r\ \ldots}$$

Ex. How many different words can be formed with all the letters of the word *dissimilar*.

Here out of the 10 letters 3 are *i*'s and 2 are *s*'s. Thus the required number of words $= \frac{10}{3\ \ 2} = 302{,}400.$

63. *To find the number of permutations of n things taken r together when each thing may be repeated any number of times in any arrangement.*

Here we have to fill up r places with n different things each of which may be used as often as r times.

The first place may be filled up in n ways, and when it has been done, the second place may again be filled up in n ways, since there is no restriction to repeat the same thing. Therefore the first two place can be filled up in $n \times n$ or n^2 ways. Next, the third place can also be filed up in n ways, and therefore the first three places can be filled up in n^3 ways.

Proceeding in this manner we shall get the number of ways to fill r places equal to n^r.

Cor The total number of permutations of n different things taken not more than r at a time, when any of the n letters may be refeated.

$$= n + n^2 + n^3 + \ldots\ldots + n^r = \frac{n(n^r - 1)}{n - 1}.$$

Ex. In how many ways can 7 oranges be destributed among 3 children when a child may get any number ?

The first orange can be given away in 3 ways as also the second. Thus first two oranges can be given away in 3^2 ways, and so on. Hence the total number of ways $=3^7=2187$.

64. *To find the number of selections of* n *things taken some or all of them together.*

Here there are two possibilities connected with each thing, *viz* ; it may be taken or rejected. Now either of the possibilities of any one thing is to be associated with either of the possibilities of any of the others. Thus the number of disposing away the n things is

$2\times2\times2\times$.....to n factors $=2^n$.

But this number includes the case in which all the n things may be rejected. Hence neglecting this one possibility, the required number of selections is 2^n-1.

Ex. In how many ways can a child make his selection from 6 balls when he is at liberty to select any number ?

The child has to select some or all of the 6 balls. He can, therefore, do so in $2^6-1=63$ ways.

Or thus. The child can select one, two, three..... balls at a time ; therefore the total number of selections

$$={}^6C_1+{}^6C_2+{}^6C_3+{}^6C_4+{}^6C_5+{}^6C_6$$
$$=6+15+20+15+6+1=63.$$

65. *To find the number of selections of* p+q *things taking some or all of them together when* p *of the things are alike of one kind and* q *are alike of a second kind.*

Here the p things can be disposed of in $p+1$ ways : since they may all be rejected or 1, 2, 3.....p of them may be taken. Also the q things can be disposed of in $q+1$ ways. Hence the total number of ways of disposing of $p+q$ things is $(p+1)(q+1)$.

But this includes the possibility of rejecting all the $p+q$ things. Hence the total number of selections is $(p+1)(q+1)-1$.

Cor. The total number of selections of $p+q+r$... things taking some or all of them together when p of

them are alike of one kind, q alike of a second kind, r alike of a third kind, and so on, is $(p+1)(q+1)(r+1)\ldots-1$.

Ex. How many numbers can be formed with one or all of the digits 2, 2, 2, 3, 3, 4, 4, 4, 4, 5.

Here three digits are 2's, two are 3's, four are 4's and one is 5. Therefore the required number

$$= (3+1)(2+1)(4+1)(1+1)-1$$
$$= 119.$$

EXERCISE XXIV

1. How many words can be framed with the letters of the word *annual* ?

2 Find the number of words framed with the letters of the word *possibilities.*

3. There are flags of n different colours ; how many signals can be given when each signal requires r flags, there being r flags of each colour ?

4 In how many ways can 35 copies of a book be arranged on 5 shelves for 7 books each ?

5. A man has 7 friends : in how many ways can he invite one or more of them to dinner ?

6 A letter lock has three rings each marked with 12 different letters ; find how many unsuccessful attempts can be made to open it ?

7. How many arrangements of 4 letters each can be made with the letters of the word *independence* ?

There are 9 different letters ; one letter is repeated twice, second thrice, third four times. Required arrangements can be made by taking (*i*) All alike (*ii*) 3 alike and one different. (*iii*) 2 pairs of alike letters, (*iv*) 2 alike and 2 unlike, (*v*) all unlike. These give rise to permutations :

(*i*) ${}^4P_4$, (*ii*) ${}^2C_1 \times {}^5C_1 \times \binom{4}{3}$, (*iii*) ${}^3C_2 \times \binom{4}{2,2}$,

(*iv*) ${}^3C_1 \times {}^5C_2 \times \binom{4}{2}$, (*v*) ${}^6P_4$. The sum of these is the required number]

8 There are 4 books on Algebra, 3 on Geometry and 5 on Calculus In how many ways can you arrange

these books on a shelf with the restriction that books on the same subject are always together ?

9. In how many ways can a committee of 4 Hindus, 3 Muslims and 2 Sikhs be formed from 10 Hindus, 5 Muslims and 3 Sikhs ?

10 In how many permutations of the letters of the word *combined* no two vowels will be together ?

11. Eight papers are set in an examination, two of them in Mathematics. In how many different orders can the papers be given, provided that the two Mathematics papers are not successive ? (P. U. 1928)

12. How many different words can be formed out of the letters of the word *Allahabad* ?

13. If you have 5 different pads of paper, 7 different pens and 3 different inks ; in how many ways can you begin to write ?

14. For 5 posts applications are received from 10 B. A's and 7 M. A's. In how many ways can the selection be made when at least one M. A. is to be taken ?

15. In a question paper there are 6 questions and each question has an alternative. Find how many selections of one or more questions can be made by a candidate.

(One question can be selected in ${}^6C_1 \times 2$ ways : two in ${}^6C_2 \times 2^2$ ways ; three in ${}^6C_3 \times 2^3$ ways ; and so on)

16. Show that the product of r consecutive integers is divisible by $\lfloor r$. (P. U. 1916)

17. A gentleman invites his $m+n$ friends to dinner and places m of them at one round table and the remaining n at another round table. Show that the number of ways in which he can arrange them among themselves is $\frac{\lfloor m+n}{m\,n}$.

18. Find the number of ways in which 6 different books may be distributed among 4 students when a student may get any number ?

19. There are 16 clerks in an office, of which 5 are junior. In how many ways may 4 of them be called to work so as to include at least one junior ?

20. Prove from the first principles that

$${}^nP_r = {}^{n-1}P_{r+1} \cdot {}^{n-1}P_{r-1}.$$ (P. U. 1923)

21. How many quadrilaterals of different areas can be formed with 7 different lines ?

22. In the first year and second year Mathematics classes there are respectively 39 and 43 students. In how many ways can they be seated in a row for examination so that no two first year students may be together ?

23. How many different cambinations and how many different permutations can be made with the letters of the word *suspects* taken 3 at a time ?

(Combinations with one s are ${}^6C_3$, with two s's ${}^5C_1$ and with three s's only one. Permutations respectively are ${}^6C_3 \times \frac{3}{2}$, ${}^5C_1 \times \frac{3}{2}$ and 1.)

24. From a paper of 8 questions how many selections of one or more questions can be made ?

25. The date-sheet of an examination of 7 papers has leaked out ; in how many ways can it be charged ?

26. A company of 60 men is to be selected from a regiment of 1,000 ; find the number of ways in which it can be done so that the same 10 men may be always included ?

27. A first year student has to take up 4 out of the 8 subjects. English being compulsory he is bound to take up only one of the three subjects Biology, Mathematics and Persian. In how many ways can he make his selection ?

28. How many selections can be made from 12 plantains, 6 oranges and 5 apples taking at least one of each kind of fruit ?

29. Show that the number of premutations of n things taken m together in which p particular things always occur is ${}^{n-p}P_{m-p} \times {}^nP_p$.

30. In how many ways can 10 apples be distributed among 4 children so that no child shall receive less than 3 apples ?

(The distribution can be effected thus : (*i*) 4, 4, 4, 4 ; (*ii*) 3, 4, 4, 5 : (*iii*) 3, 3, 4, 6 ; (*iv*) 3, 3, 5, 5 ; (*v*) 3, 3, 3, 7 ; Now (*i*) gives only one way, (*ii*) ${}^4C_1 \times {}^3C_1$ ways (*iii*) ${}^4C_2 \times {}^2C_1$ ways : (*iv*) ${}^4C_2$ ways ; (*v*) ${}^4C_1$ ways.)

31. In how many ways can a cricket eleven be selected from 25 players, 6 of them being always included and 5 always excluded ?

32. Prove that in ${}^{2n}C_n$ the number of combinations in which a particular thing occurs is equal to the number in which it does not occur.

33. Out of $2n$ men who have to sit down half on each side of a table, p particular men desire to sit on one side and q on the other ; find the number of ways in which this can be done.

(On one side $n-p$ men from $2n-p-q$; remaining on the other. Then the arrangements on each side are $\underline{n}$).

34. Show that the greatest number of combinations that can be formed with $2n$ things, each containing the same number is always double the greatest number that can be formed with $2n-1$ things.

35. If in the town there be four boy schools and three girl schools, find in how many ways a man can send his two sons and three daughters to school ?

[First boy school may be utilised in 2 ways: first two i 2^2 ways : and so on. Boy schools give the number 2^4. Similarly the girl schools give 3^3. Hence the required number is $2^4 \times 3^3$].

36. In how many ways can a cricket eleven be chosen from 13 members of whom 4 can bowl, so as to include at least 2 bowlers ?

37. The number of combinations of n things taken 5 together in which 3 particular things occur is 21 ; find the number of combinations of them taken 6 together in which 4 given things occur.

38. In how many ways can an examiner assign a total of 30 marks to 8 questions without giving less than 2 marks to any question ?

[Proceed as in question 30].

39. In how many ways can the 4 suits of playing cards be dealt to 4 players, thirteen to each, so that every one may have a card of each number ?

40. Show that the number of combinations of $p+q+r$ things taken some or all of them together, with the restriction that p of them are alike of one kind, q of them alike of a second kind and the rest all different, is $(p+1)(q+1)2^r-1$.

CHAPTER X

Binomial Theorem

Positive Integral Index.

66. Definitions. An algebraical expression of two terms is called a *binomial*.

$$x+a,\quad 5x^2+y,\quad 3x-by^2,\quad alp-bmq$$

are all binomials.

The formula

$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_{n-1}xa^{n-1} + a^n$,

where n is a positive integer, is known by the name of **Binomial Theorem.**

By the help of the Binomial Theorem we can easily find the continued product of a binomial by itself any number of times or we can find the root of a binomial.

67. *To prove the Binomial Theorem when* n *is a positive integer.*

By actual multiplication

$$(x+a_1)(x+a_2) = x^2 + (a_1+a_2)x + a_1a_2\ ;$$

$$(x+a_1)(x+a_2)(x+a_3) = x^3 + (a_1+a_2+a_3)x^2 + (a_2a_3+a_3a_1+a_1a_2)x + a_1a_2a_3\ ;$$

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4) = x^4 + (a_1+a_2+a_3+a_4)x^3 + (a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)x^2 + (a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)x + a_1a_2a_3a_4\ .$$

In each of these results we observe that :—

(*i*) The number of terms in each of the products is one more than the number of binomial factors multiplied together.

(*ii*) The power of x in the first term is the same as the number of binomial factors ; and the power of x decreases by unity in all the succeeding terms.

(*iii*) The co-efficient of the first term is unity ; that of the second term is the sum of all the second terms of the binomial factors ; that of the third term is the sum of all the products of these second terms taken two at a time ; that of the fourth term is the sum of all the products of these second terms taken three at a time ; and so on ; the last term is the product of all these second terms.

Proceeding in the light of above observations, we infer that the product of n factors

$$(x+a_1)(x+a_2)(x+a_3).....(x+a_n)$$
$$=x^n+S_1x^{n-1}+S_2x^{n-2}+..,+S_rx^{n-r}+...+S_n.....(1)$$

where S_1 = the sum of all the second terms a_1, $a_2...a_n$ of n factors.

S_2 = the sum of all the products of the second terms $a_1, a_2...,a_n$ taken two at a time.

.....

S_r = the sum of all the products of the second terms $a_1, a_2,...a_n$ taken r at a time

. . . . . .

S_n = the product of all the second terms, $a_1, a_2,...,a_n$.

Now evidently

the *number of terms* in $S_1 = {}^nC_1$

" " " " " $S_2 = {}^nC_2$

. . . . . .

" " " " " $S_r = {}^nC_r$

. . . . . .

If we put $a_1=a_2=a_3=...=a_n=a$, then

$(x+a_1)(x+a_2)(x+a_3)...(x+a_n)=(x+a)^n$;

and $S_1={}^nC_1\times a$; $S_2={}^nC_2\times a^2$;

$S_3={}^nC_3\times a^3$; . . . . . .

. . . . ; $S_r={}^nC_r\times a^r$;

. . . . ; $S_n=a^n$.

Hence substituting all these values in (1) we get

$$(x+a)^n=x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+...+{}^nC_rx^{n-r}a^r+...+a^n,$$

which is the Binomial Theorem.

Note. This may also be written as

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{2}x^{n-2}a^2$$
$$+\frac{n(n-1)(n-2)}{3}x^{n-3}a^3+...+a^n.$$

68. A second proof by Mathematical *Induction.*

Multiply both sides of

$$(x+a)^n=x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+...+{}^nC_rx^{n-r}a^r+...+a^n$$

by $x+a$. Then

$$(x+a)^{n+1}=x^{n+1}+(1+{}^nC_1)x^na+({}^nC_1+{}^nC_2)x^{n-1}a^2+.....$$
$$+({}^nC_{r-1}+{}^nC_r)x^{n-r+1}a^r+...+a^{n+1}.$$

But since $1+{}^nC_1=1+n={}^{n+1}C_1$;

${}^nC_1+{}^nC_2={}^{n+1}C_2$;

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$, and so on.

$\therefore \quad (x+a)^{n+1} = x^{n+1} + {}^{n+1}C_1 x^n a + {}^{n+1}C_2 x^{n-1} a^2 + \ldots\ldots$
$+ {}^{n+1}C_r x^{n-r+1} a^r + \ldots + a^{n+1}.$

Thus we see that if the formula is true for n, then it is also true for $n+1$.

But we know that by actual multiplication,

$(x+a)^2 = x^2 + 2xa + a^2$
$= x^2 + {}^{2}C_1 xa + a^2$;

$(x+a)^3 = x^3 + 3x^2 a + 3xa^2 + a^3$
$= x^3 + {}^{3}C_1 x^2 a + {}^{3}C_2 xa^2 + a^3$;

and $(x+a)^4 = x^4 + 4x^3 a + 6x^2 a^2 + 4xa^3 + a^4$
$= x^4 + {}^{4}C_1 x^3 a + {}^{4}C_2 x^2 a^2 + {}^{4}C_3 xa^3 + a^4.$

It means that the formula is true for $n=2$, 3 or 4 ; and therefore it must be true for $n=5$, and therefore for $n=6$, and so on.

Hence it is true for all positive integral values of n.

Note. Observe the following facts about the Binomial Theorem :—

(i) The number of terms in the expansion is one more than the power of the binomial.

(ii) The power of x goes on decreasing by unity and the power of a goes on increasing by unity, the sum of the power of x and a being always the same and equal to the power of the binomial.

(iii) The co-efficient of any term is the number of combination formed from things equal to the power of the binomial, the things taken being equal to the number of previous terms or to the power of a in that term.

Cor. 1. Interchanging x and a the Binomial Theorem may be written as

$(a+x)^n = a^n + {}^{n}C_1 a^{n-1} x + {}^{n}C_2 a^{n-2} x^2 + \ldots\ldots + {}^{n}C_r a^{n-r} x^r$
$+ \ldots\ldots + x^n$

Here put $a=1$. Then

$(1+x)^n = 1 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots\ldots + {}^{n}C_r x^r + \ldots\ldots + x^n \ldots\ldots (A)$

This form of the Binomial Theorm is very convenient. Every binomial can be put in this form and expanded.

For $(a+x)^n = a^n \left(1 + \frac{x}{a}\right)^n$

$= a^n \left(1 + {}^{n}C_1 \frac{x}{a} + {}^{n}C_2 \frac{x^2}{a^2} + \ldots\ldots + {}^{n}C_r \frac{x^r}{a^r} + \ldots\ldots + \frac{x^n}{a^n}\right).$

Cor 2. In (A) change x into $-x$. Then

$$(1-x)^n = 1 + {}^nC_1(-x) + {}^nC_2(-x)^2 + \ldots\ldots + {}^nC_r(-x)^r + \ldots\ldots + (-x)^n$$
$$= 1 - {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + (-1)^r\,{}^nC_rx^r + \ldots\ldots\ldots + (-1)^nx^n.$$

Thus in the expansion of $(1-x)^n$ the terms are alternately positive and negative, the last terms being positive if n is even and nagative if n is odd.

Cor. 3. The Binomial Theorem is an identity and hence true for all values of x.

69. The $(r+1)$th term in the expansion is called the *general term* of the Binomial Theorm ; for by giving suitable value to r, any term of the expansion can be found.

Evidently in the expansion of $(x+a)^n$

$$T_{r+1} = {}^nC_rx^{n-r}a^r ;$$

and in the expansion of $(1+x)^n$

$$T_{r+1} = {}^nC_rx^r.$$

Ex. 1. Expand $(3x+2b)^7$.

$$(3x+2b)^7 = (3x)^7 + {}^7C_1(3x)^6(2b) + {}^7C_2(3x)^5(2b)^2 + {}^7C_3(3x)^4(2b)^3 + {}^7C_4(3x)^3(2b)^4 + {}^7C_5(3x)^2(2b)^5 + {}^7C_6(3x)(2b)^6 + (2b)^7.$$

$$= 2187x^7 + 7.729.2x^6b + \frac{7.6}{2}.243.4x^5b^2 + \frac{7.6.5}{.2.3}.81.8x^4b^3 + \frac{7.6.5.4}{2.3.4}.27.16x^3b^4 + \frac{7.6.5.4.3}{2.3.4.5}.9.32x^2b^5 + \frac{7.6\,5\,4.3.2.}{2.3.4\;5.6.}.3.64xb^6 + 128b^7$$

$$= 2187x^7 + 10206x^6b + 20412x^5b^2 + 22680x^4b^3 + 15120x^3b^4 + 6048x^2b^5 + 1344xb^6 + 128b^7.$$

Ex. 2. Find the 8th term in the expansion of $(2x-3y)^{13}$.

$$T_8 = {}^{13}C_7(2x)^{13-7}(-3y)^7 = {}^{13}C_6(2x)^6(-3y)^7$$
$$= -\frac{13.12.11.10.9.8}{1.2.3.4.5.6}.2^6.3^7x^6y^7$$
$$= -1716.2^6.3^7x^6y^7 = -240185088x^6y^7.$$

Ex. 3. Find (*i*) the middle term and (*ii*) the term independent of x in $\left(x-\frac{1}{x^2}\right)^{12}$.

(*i*) The number of terms in the expansion being 13, the middle term is the 7th. Thus

$$T_7={}^{12}C_6x^{12-6}\left(-\frac{1}{x^2}\right)^6$$

$$=\frac{12.11.10.9.8.7}{1.2.3.4.5.6}x^6.\frac{1}{x^{12}}$$

$$=924.\frac{1}{x^6}.$$

(*ii*) Suppose that $(r+1)$th term does not contain x. Then since

$$T_{r+1}={}^{12}C_rx^{12-r}\left(-\frac{1}{x^2}\right)^r$$

$$={}^{12}C_r(-1)^rx^{12-3r},$$

we must have $12-3r=0$, *i. e.*, $r=4$.

Thus the 5th term is independent of x. And

$$T_5={}^{12}C_4(-1)^4=\left|\begin{matrix}12\\4\quad 8\end{matrix}\right.=495.$$

Ex. 4. What is the coefficient of $\frac{1}{y^3}$ in the expansion of $\left(ay+\frac{1}{y^2}\right)^9$?

Suppose that $\frac{1}{y^3}$ occurs in the $(r+1)$th term. Then since

$$T_{r+1}={}^9C_r(ay)^{9-r}\left(\frac{1}{y^2}\right)^r={}^9C_ra^{9-r}y^{9-3r},$$

we must have $9-3r=-3$, *i. e.*, $r=4$.

Thus the 5th term has $\frac{1}{y^3}$. And then the required coefficient $={}^9C_4a^{9-4}=\left|\begin{matrix}9\\4\quad 5\end{matrix}\right.a^5=126a^5$.

Ex. 5. If the four consecutive terms of a binomial expansion be 4, 7, 7 and $4\frac{3}{8}$, find the expansion.

Let the binomial be $(1+x)$ and let the terms be rth $(r+1)$th, $(r+2)$th, $(r+3)$th.

Then ${}^nC_{r-1}x^{r-1}=4$, ${}^nC_rx^r=7$, ${}^nC_{r+1}x^{r+1}=7$ and ${}^nC_{r+2}x^{r+2}=4\frac{3}{8}$

$$\therefore \quad \frac{{}^nC_rx^r}{{}^nC_{r-1}x^{r-1}}=\frac{7}{4},\ i.\ e.,\ 4(n-r+1)x=7r \ldots\ldots\ldots(i)$$

$$\text{Also} \quad \frac{{}^nC_{r+1}x^{r+1}}{{}^nC_rx^r}=1,\ i.\ e.,\ (n-r)x=r+1 \ldots\ldots\ldots(ii)$$

$$\text{and} \quad \frac{{}^nC_{r+2}x^{r+2}}{{}^nC_{r+1}x^{r+1}}=\frac{35}{8}/7,\ i.\ e.,\ 8(n-r-1)x=5(r+2)$$

$\ldots\ldots\ldots\ldots(iii)$

Solving (i), (ii), and (iii) simultaneously, we get

$r=2$, $x=\frac{1}{3}$ and $n=8$.

Therefore the binomial is $(1+\frac{1}{3})^8$.

EXERCISE XXV

1. Expand the following binomials :—

(i) $(x+2a)^5$. (ii) $(1-x)^7$.

(iii) $\left(x+\frac{1}{x}\right)^6$. (iv) $\left(\frac{2x}{3}+\frac{3}{2x}\right)^5$.

(v) $(1-3x^2)^6$. (vi) $\left(x^2+\frac{2y}{x}\right)^7$.

(vii) $(a^{\frac{1}{2}}-b^{\frac{1}{2}})^4$. $(viii)$ $\left(3a+\frac{1}{3a}\right)^7$.

(ix) $\left(\frac{x^2}{2}-\frac{2}{x^2}\right)^8$.

2. Find the value of :—

(i) $(1+\sqrt{2})^5+(1-\sqrt{2})^5$.

(ii) $(\sqrt{3}+x)^7-(\sqrt{3}-x)^7$.

(iii) $(x-\sqrt{x^2-a^2})^5+(x+\sqrt{x^2-a^2})^5$.

3. What is the 5th term in the expansion of $\left(\frac{2}{x}+3x\right)^{10}$?

4. Find the 9th term of $\left(1-\frac{x}{\sqrt{3}}\right)^{12}$.

5. Write down the 20th term of $\left(\frac{p}{\sqrt{10}}+\frac{1}{p}\right)^{25}$.

6. Find the term independent of a in $\left(\frac{x}{a}+\frac{a^3}{3x}\right)^{10}$.

7. What is the middle term of $(3y-\frac{1}{3}x)^{14}$?

8. Find the two middle terms of $\left(\frac{x}{\sqrt{2}}+\frac{\sqrt{3}}{a}\right)^{9}$.

9. What is the coefficient of x^3 in $\left(2x-\frac{1}{3x}\right)^{13}$?

10. Find the coefficient of x^{18} in $(a^4-bx^3)^{10}$.

11. Find the coefficients of x^{-10} and x^9 in $\left(x^4+\frac{1}{x^2}\right)^{8}$.

12. Is there any absolute term in $\left(\frac{3}{2}y^2-\frac{1}{3y}\right)^{9}$? If so, find it.

13. Show that the coefficient of $(p+1)$th term in $(1+x)^{n+1}$ is the same as the sum of the coefficients of pth and $(p+1)$th terms in $(1+x)^n$.

14. Show that the coefficient of x^n in $(1+x)^{2n}$ is double that of x^n in $(1+x)^{2n-1}$.

15. Show that the coefficients of x^{2r} and $\frac{1}{x^{12}}$ in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ are equal in magnitude but opposite in sign.

16. Find n if the coefficients of 2nd, 3rd and 4th terms of $(1+x)^n$ are in A. P.

17. Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{\lfloor n} .2^n.x^n$.

18. If the term containing x occurs in the expansion of $\left(x+\frac{1}{x}\right)^n$, show that its coefficient is

$$\frac{\lfloor n}{\lfloor \frac{1}{2}(n-r) \cdot \lfloor \frac{1}{2}(n+r)}.$$

19. Of which expansion are 14, 84, 280 and 560 four consecutive terms ?

20. If three consecutive coefficients of $(1+x)^n$ are 6, 15 and 20 ; find n.

70. The greatest coefficient in the expansion of $(1+x)^n$.

The coefficient of the $(r+1)$th term in the expansion of $(1+x)^n$ is ${}^nC_r$. We have to find the value of r for which ${}^nC_r$ is greatest.

Now we know (Art. 49, Ex. 8) that if n be even ${}^nC_r$ is greatest when $r=\frac{n}{2}$, and that if n be odd ${}^nC_r$ is greatest when $r=\frac{n-1}{2}$ or $\frac{n+1}{2}$.

Hence when n is even, the greatest coefficient in ${}^nC_r$ is ${}^nC_{\frac{n}{2}}$ and when n is odd it is ${}^nC_{n-1}$ or ${}^nC_{n+1}$, both of them being equal.

71. The greatest term in the expansion of $(a+x)^n$.

Let T_r and T_{r+1} be the rth and $(r+1)$th terms respectively.

Then $$T_r={}^nC_{r-1}a^{n-r+1}x^{r-1}$$

and $$T_{r+1}={}^nC_r a^{n-r}x^r$$

$$\therefore \quad \frac{T_{r+1}}{T_r}=\frac{{}^nC_r}{{}^nC_{r-1}}\cdot\frac{x}{a}$$

$$=\frac{n-r+1}{r}\cdot\frac{x}{a}=\left(\frac{n+1}{r}-1\right)\frac{x}{a}$$

Thus we see that $T_{r+1} >, =,$ or $< T_r$

according as $\left(\frac{n+1}{r} - 1\right)\frac{x}{a} >, =,$ or $< 1,$

i.e., according as $\frac{n+1}{r} - 1 >, =,$ or $< \frac{a}{x}$

i.e., according as $\frac{n+1}{r} >, =,$ or $< \frac{a}{x} + 1$

i.e., according as $\frac{n+1}{\frac{a}{x}+1} >, =,$ or $< r$

i.e., according as $r <, =,$ or $> \frac{n+1}{\frac{a}{x}+1}$.

Now two cases may arise ; *firstly* $\frac{n+1}{\frac{a}{x}+1}$ may be a whole number, *secondly* it may be fractional.

(*i*) If $\frac{n+1}{\frac{a}{x}+1}$ is an integer, let it be p. Then for all values of r from 1 to $p-1$, the value of $\left(\frac{n+1}{r} - 1\right)\frac{x}{a}$ is greater than unity and $T_{r+1} > T_r$. It means that the terms go on increasing. If $r = p$, then $\left(\frac{n+1}{r} - 1\right)\frac{x}{a} = 1$.

Then $T_r = T_{r+1}$, *i. e.*, $T_p = T_{p+1}$, These two are the greatest terms.

(*ii*) If $\frac{n+1}{\frac{a}{x}+1}$ is fractional, let its two parts be q and s, where q is an integer and s a proper fraction. Then $r < q + s$: therefore the greatst integral value of r is q. Hence T_{q+1} is the greatest term.

Ex. What is the greatest term of $(3-5x)^{11}$ when $x=\frac{1}{5}$.

We have *numerically* $\frac{T_{r+1}}{T_r}=\frac{11-r+1}{r}\cdot\frac{5x}{3}$

$$=\frac{12-r}{r}\cdot\frac{5}{3}\times\frac{1}{5}=\frac{12-r}{3r}.$$

Hence $T_{r+1}>,=,$ or $<T_r$ according as $\frac{12-r}{3r}>,=,$ or <1,

i. e., according as $12-r>,=,$ or $<3r$,

i. e., according as $r<,=,$ or >3.

It means that if $r=1$ or 2, T_{r+1} is greater than T_r; when $r=3$, then $T_r=T_{r+1}$; and when $r=4$, 5.....then T_{r+1} is less than T_r.

Hence the 3rd and 4th terms are *numerically* equal and greater than any other term.

The value of the greatest term $={}^{11}C_2\,3^9\,(5x)^2$

$$=\frac{11\times10}{2}\times3^9=55\times3^9.$$

72. Properties of the Binomial Co-efficients.

The successive co-efficients of various terms in the expansion of $(1+x)^n$ or $(a+x)^n$ are

$$1,\ n,\ \frac{n(n-1)}{2},\ \frac{n(n-1)(n-2)}{3},\ \ldots\ldots\frac{n}{n-1},\ \frac{n}{n}.$$

or ${}^nC_0,\ {}^nC_1,\ {}^nC_2\ldots\ldots\ldots\ldots{}^nC_{n-1},\ {}^nC_n$.

These are called the *Binomial co-efficients.* We shall now establish the three important properties of these co-efficients.

I. *The co-efficients of terms equidistant from the beginning and the end are equal.*

Let us consider the co-efficients of the $(r+1)$th term from the beginning and the $(r+1)$th term from the end.

The co-efficient of the $(r+1)$th term from the beginning is ${}^nC_r$.

The total number of terms in the expansion being $n+1$, the $(r+1)$th term from the end has $(n+1)-(r+1)$ or $n-r$ terms before it, *i.e.*, the $(r+1)$th term from the end is the $(n-r+1)$th term from the beginning, and therefore its co-efficient is ${}^nC_{n-r}$

But we know that ${}^nC_r = {}^nC_{n-r}$.

Hence the co-efficients of the $(r+1)$th terms from the begining and end are equal.

II. *The sum of the Binomial Co-efficients is* 2^n.

In the Binomial Theorem

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n$,,

put $x=1$. Then

$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_{n-1} + {}^nC_n =$ the required sum.

III. *The sum of the co efficients of even terms is equal to the sum of the co-efficients of odd terms.*

In the Binomial Theorem

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + {}^nC_5x^5 + \ldots + {}^nC_rx^r \ldots\ldots\ldots\ldots + {}^nC_nx^n$

put $x=-1$. Then

$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \ldots\ldots\ldots + (-1)^r{}^nC_r + \ldots\ldots + (-1)^n$

$\therefore \quad {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots$

Note. The Binomial Co-efficients are often briefly written as $C_0, C_1, C_2, \ldots\ldots\ldots C_n$.

Ex. 1. Find the value of

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots\ldots + \frac{C_n}{n+1}$$

We have $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots\ldots + \frac{C_n}{n+1}$

$= 1 + \frac{1}{2}.n + \frac{1}{3}.\frac{n(n-1)}{2} + \ldots\ldots\ldots + \frac{1}{n+1}\frac{n}{n}$.

$= \frac{1}{n+1}\left\{ n+1 + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3.2} + \ldots + \frac{n}{n} \right\}$

$= \frac{1}{n+1}\left[\left\{ 1 + (n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3} + \ldots + \frac{n+1}{n+1} \right\} - 1 \right]$

$= \frac{1}{n+1}\left[(1+1)^{n+1} - 1 \right]$

$$= 2^{n+1} - 1.$$

Ex. 2. Find the sum of the squares of the Binomial Co-efficients.

We have

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots\ldots C_nx^n \ldots (1)$$

Also the same may be written as

$$(1+x)^n = C_nx^n + C_{n-1}x^{n-1} + \ldots + C_1x + C_0$$
$$= C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_{n-1}x + C_n \ldots (2)$$

since the co-efficients from the beginning and the end are equal.

Multiply (1) and (2) together.

$$\therefore \quad (1+x)^{2n} = (C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2)x^n + (C_0C_1 + C_1C_2 + C_2C_3 + \ldots)x^{n+1} + (C_0C_2 + C_1C_3 + C_2C_4 + \ldots\ldots)x^{n+2} + \ldots\ldots\ldots\ldots$$

In this identity we see that the required sum is the co-efficient of x^n, and therefore it must be equal to the co-efficient of x^n in the expansion of $(1+x)^{2n}$, which is ${}^{2n}C_n$.

Hence $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{\lfloor 2n}{(\lfloor n)^2}$.

Ex 3. Prove that if n be any positive integer, the integral part of $(2+\sqrt{3})^n$ is an odd number.

The intergal part of $2+\sqrt{3}$ is seen to be 3, an odd number. Now we have to find the integrel part of $(2+\sqrt{3})^n$.

Let the integral part of $(2+\sqrt{3})^n$ be I and the decimal part F, so that

$$I + F = (2+\sqrt{3})^n = 2^n + {}^nC_1\, 2^{n-1}\sqrt{3} + {}^nC_2\, 2^{n-2}(\sqrt{3})^2 + \ldots\ldots\ldots (1)$$

Now $(2-\sqrt{3})^n$ is positive, since $2-\sqrt{3}$ is positive.

$$\therefore \quad G = (2-\sqrt{3})^n = 2^n - {}^nC_1 2^{n-1}\sqrt{3} + {}^nC_2 2^{n-2}(\sqrt{3})^2 \ldots\ldots (2)$$

where G denotes $(2-\sqrt{3})^n$

Adding (1) and (2), we get

$$I + F + G = 2(2^n + {}^nC_2 2^{n-2}.3 + {}^nC_4 2^{n-4}.3^2 + \ldots\ldots),$$
$$= \text{an even integer} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\therefore \quad F + G = \text{an even integer} - I = \text{an integer.}$$

Since F and G are each of them less than unity, and $F+G$ is less than 2, and $F+G$ is an integer, therefore, $F+G=1$.

Now from(3) $I+F+G=I+1=$ an even integer.
$\therefore$ I is an odd integer.

Exercise XXVI.

1. Find which term is greatest in the expansion of
(*i*) $(1+x)^8$ when $x=\frac{1}{3}$. (*ii*) $(2+3x)^{12}$ when $x=\frac{5}{6}$.
(*iii*) $(2x+5y)^{34}$ when $x=3$, $y=2$.
(*iv*) $(\frac{3}{4}a-\frac{2}{3}b)^n$ when $n=19$, $a=8$, $b=3$.

2. Find the greatest term in
(*i*) $(\frac{1}{3}+x)^9$ when $x=\frac{1}{3}$.
(*ii*) $(\frac{1}{3}m-\frac{1}{2}n)^6$ when $m=9$, $n=4$.
(*iii*) $(p+3q)^n$ when $p=2$, $q=\frac{2}{3}$, $n=14$.

3. The coefficients of $(r+1)$th and $(r+3)$th terms in the expansion of $(1+x)^{2n}$ are equal; find n.

4. What are the rth term from the beginning, and the rth term from the end and the middle term of $\left(x-\frac{1}{x}\right)^{2n}$?

5. Find the value of :—
(*i*) ${}^{12}C_1+{}^{12}C_2+{}^{12}C_3+\ldots+{}^{12}C_{12}$.
(*ii*) ${}^{19}C_1+{}^{19}C_2+{}^{19}C_3+\ldots+{}^{19}C_{19}$.
(*iii*) ${}^{22}C_1+{}^{22}C_2+{}^{22}C_3+\ldots+{}^{22}C_{21}$.

6. Find the value of :—
(*i*) $C_1+2C_2+3C_3+\ldots+nC_n$.
(*ii*) $C_0+2C_1+3C_2+\ldots+(n+1)C_n$.
(*iii*) $C_0C_1+C_1C_2+C_2C_3+\ldots+C_{n-1}C_n$.
(*iv*) $C_0C_2+C_1C_3+C_2C_4+\ldots+C_{n-2}C_n$.
(*v*) $C_0C_3+C_1C_4+C_2C_5+\ldots+C_{n-3}C_n$.
(*vi*) $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots+C_{n-r}C_n$.
(*vii*) $\frac{C_1}{C_0}+\frac{2C_2}{C_1}+\frac{3C_3}{C_2}+\ldots+\frac{nC_n}{C_{n-1}}$.

7. Expand $(1+2x-x^2)^4$.

8. Expand $(1+x+x^2)^5$.

9. Find the coefficient of x^5 in $(1+x+x^2+x^3)^n$.

10. If A and B are respectively the sums of the odd and even terms in the expansion of $(x+a)^n$, then show that
(*i*) $A^2-B^2=(x^2-a^2)^n$.
and (*ii*) $4AB=(x+a)^{2n}-(x-a)^{2n}$.

11. Prove that if n be any positive integer, the integral part of $(3+\sqrt{5})^n$ is an odd number.

CHAPTER XI

Binomial Theorem.

Any Index.

73. In the last chapter it was proved that when n is a positive integer,

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + x^n$$
$$= 1 + nx + \frac{n(n-1)}{\underline{2}} x^2 + \ldots + x^n,$$

there being $n+1$ terms on the right-hand side.

Now if n be a fraction or a negative number, none of the quantities n, $\frac{n(n-1)}{\underline{2}}$, $\frac{n(n-1)(n-2)}{\underline{3}}$ can be an integer, so that ${}^nC_1$, ${}^nC_2$, ${}^nC_3$,...always representing some positive integers, are meaningless in this case. Thus we shall state

The Binomial Theorem when n *is any negative number, integral or fractional* :—

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{2}} x^2 + \frac{n(n-1)(n-2)}{\underline{3}} x^3 + \ldots \text{to}$$

infinity, provided that the numerical value of x is less than unity.

The series on the right-hand side of this statement extends to infinity because the quantity $n+1$, giving us the total number of terms in the expansion can never be a positive integer ; and as long as $n+1$ is not a positive integer, the number of terms, cannot be finite.

For, in order that the number of terms be finite, *i.e.*, the series may stop at some stage, $n-r+1$ must vanish, *i. e.*, $r=n+1$. Now, r is always a positive integer. Therefore $n+1$ or n must always be a positive integer if the series is to stop. Hence, the expansion must extend to infinity then n is not a positive integer.

The important point to be noted in this case, is the *condition*, that *x should be less than unity*. This is so in order that the series on the right hand side may be convergent. For, if it is not so, absurd results follow. For example by the above statement

$$(1-x)^{-2}=1+(-2)(-x)+\frac{(-2)(-2-1)}{2}(-x)^2+\ldots\ldots$$

$$=1+2x+3x^2+4x^3+\ldots\ldots$$

If we put $x=2$ on both sides, we get

$$1=1+2+3+4+\ldots\ldots$$

which is absurd. It should be noted that for the numerical values of $x \geqslant 1$, the series becomes divergent.

Note 1. By this theorem can be expanded the binomial $(a+x)^n$, n being a fraction or a negative number, even when a or x is not less than unity. For a and x being unequal

$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ or $x^n\left(1+\frac{a}{x}\right)^n$ according as a is greater or less than x.

Note 2. The $(r+1)$th, *i. e.*, the general term here is

$$\frac{n(n-1)\ldots(n-r+1)}{r}.x^r \text{ and not } {}^nC_r x^r.$$

74. To find the general term in the expansions of (*i*) $(1+x)^{-n}$ (*ii*) $(1-x)^{-n}$.

(*i*) $$T_{r+1}=\frac{-n(-n-1)(-n-2)\ldots(-n-r+1)}{r}x^r$$

$$=(-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r}x^r$$

(*ii*) Putting $-x$ for x in the T_{r+1} of (*i*), we have

$$T_{r+1}=(-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r}(-x)^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\ldots(n+r-1)}{r}(-x^r)$$

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{r}x^r.$$

Ex. Write down the general term in the expansions of (*i*) $(1+x)^{-\frac{3}{2}}$ and (*ii*) $(1-x)^{-\frac{1}{2}}$.

(*i*) $$T_{r+1}=(-1)^r\frac{3.5.7.9\ldots(r+1)}{2^r\,r}.x^r$$

$$(ii)\quad T_{r+1}=\frac{1.3.5\ldots(r-1)}{2^r\ \ r}x^r$$

Note. In case *(i) viz.*, $(1+x)^{-n}$ above, the terms in the expansion are alternately positive and negative; and in case *(ii) viz.*, $(1-x)^{-n}$, all the terms will be positive, if x and n are positive.

Ex. 1. Expand $\frac{1}{(3+2x^2)^{\frac{4}{5}}}$ and find the general term.

$$\frac{1}{(3+2x^2)^{\frac{4}{5}}}=\frac{1}{3^{\frac{4}{5}}\left(1+\frac{2x^2}{3}\right)^{\frac{4}{5}}}$$

$$=\frac{1}{3^{\frac{4}{5}}}\left(1+\frac{2x^2}{3}\right)^{-\frac{4}{5}}$$

$$=\frac{1}{\sqrt[5]{81}}\left(1+(-\tfrac{4}{5})\frac{2x^2}{3}+\frac{(-\frac{4}{5})(-\frac{4}{5}-1)}{2}\left(\frac{2x^2}{3}\right)^2\right.$$

$$\left.+\ldots\ldots\ldots\right)$$

$$=\frac{1}{\sqrt[5]{81}}\left(1-\frac{8x^2}{15}+\frac{8x^4}{25}-\ldots\ldots\ldots\right).$$

$$T_{r+1}=\frac{1}{\sqrt[5]{81}}\frac{(-\frac{4}{5})(-\frac{4}{5}-1)(-\frac{4}{5}-2)\ldots(-\frac{4}{5}-r+1)}{r}\left(\frac{2x^2}{3}\right)^r$$

$$=(-1)^r\cdot\frac{1}{3^{\frac{4}{5}}}\frac{\frac{4}{5}\cdot\frac{9}{5}\cdot\frac{14}{5}\ldots\ldots(r-\frac{1}{5})}{r}\frac{2^r}{3^r}x^{2r}$$

$$=(-1)^r\frac{4.9.14\ldots\ldots(5r-1)}{5^r\ \ r}\cdot\frac{2^r}{3^{r+\frac{4}{5}}}x^{2r}.$$

Ex. 2. Expand to 5 terms and find the general term of $\sqrt{a^2-x^4}$.

$$\sqrt{a^2-x^2}=(a^2-x^2)^{\frac{1}{2}}=a\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}$$

$$= a\left\{1+\frac{1}{2}\left(-\frac{x^2}{a^2}\right)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\left(-\frac{x^2}{a^2}\right)^2\right.$$
$$+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3}\left(-\frac{x^2}{a^2}\right)^3+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4}$$
$$\left.\left(-\frac{x^2}{a^2}\right)^4+\ldots\ldots\right\}, \text{ if } x<a$$

$$= a\left\{1-\frac{1}{2}\cdot\frac{x^2}{a^2}-\frac{1}{8}\cdot\frac{x^4}{a^4}-\frac{1}{16}\frac{x^6}{a^6}-\frac{5}{128}\frac{x^8}{a^8}\right.$$
$$\left.-\ldots\ldots\right\}$$

The general terms

$$= a\,\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-r+1)}{r}\left(-\frac{x^2}{a^2}\right)^r$$

$$= a(-1)^{r-1}\,\frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\cdots\frac{2r-3}{2}}{r}(-1)^r\,\frac{x^{2r}}{a^{2r}}$$

$$= -a\,\frac{1.3.5\ldots\ldots(2r-3)}{2^r\ \ r}\left(\frac{x}{a}\right)^{2r}.$$

Question : What will be the expansion if $x>a$?

Ex. 3. Find the first negative term in the expansion of $(1+2x)^{\frac{9}{2}}$

Suppose that T_{r+1} is the first negative term, so that T_r is positive. But $T_{r+1}=\frac{\frac{9}{2}(\frac{9}{2}-1)\ldots(\frac{9}{2}-r+1)}{r}(2x)^r$

and $$T_r=\frac{\frac{9}{2}(\frac{9}{2}-1)\ldots(\frac{9}{2}-r)}{r-1}(2x)^{r-1}$$

$$\therefore\quad \frac{T_{r+1}}{T_r}=\frac{\frac{9}{2}-r+1}{r}.2x.$$

Now T_{r+1} is negative if $\frac{\frac{9}{2}-r+1}{r}.2x$ is negative, *i. e.* if $\frac{9}{2}-r+1$ is negative, *i. e.* if $r>5\frac{1}{2}$.

Hence when $r=6$ we get the first negative term. Hence the 7th term is the first negative term.

To find its value $T_7 = \frac{\frac{9}{2}(\frac{9}{2}-1)\ldots(\frac{9}{2}-6+1)}{6}(2x)^6$

$$= -\frac{9.7\ 5.3.1.3}{2^6\ \ 6}\,2^6x^6$$

$$= -\frac{315}{80}\,x^6.$$

Ex. 4. The following expansions may easily be established.

$(1+x)^{-1} = 1-x+x^2-\ldots\ldots\ldots+(-1)^r x^r+\ldots\ldots\ldots$

$(1+x)^{-2} = 1-2x+3x^2-\ldots\ldots+(-1)^r(r+1)x^r+\ldots\ldots$

$(1+x)^{-3} = 1-3x+6x^2-\ldots\ldots+(-1)^r\frac{(r+1)(r+2)}{2}x^r$

$+\ldots\ldots\ldots$

$(1-x)^{-1} = 1+x+x^2+\ldots\ldots\ldots+x^r+\ldots\ldots\ldots$

$(1-x)^{-2} = 1+2x+3x^2+\ldots\ldots+(r+1)x^r+\ldots\ldots\ldots$

$(1-x)^{-3} = 1+3x+6x^2+\ldots\ldots+\frac{(r+1)(r+2)}{2}x^r+\ldots\ldots$

These expansions are frequently used. The student would do well to commit them to memory.

Ex. 5. Find the coefficient of x^r in the expansion of $\left(\frac{1-x}{1+x}\right)^2$.

$$\left(\frac{1-x}{1+x}\right)^2 = (1-x)^2(1+x)^{-2}$$

$$= (1-2x+x^2)\{1-2x+3x^2-\ldots\ldots$$
$$+(-1)^r(r+1)x^r+\ldots\ldots\}$$

$\therefore$, the coefficient of x^r

$= (-1)^r(r+1)-2(-1)^{r-1}r+(-1)^{r-2}(r-1)$

$= (-1)^r\{r+1-2(-1)^{-1}r+(-1)^{-2}(r-1)\}$

$= (-1)^r\{r+1+2r+r-1\}$

$= (-1)^r.4r.$

Ex. 6. Find the greatest term in the expansion of $(5-4x)^{-7}$, when $x=\frac{1}{2}$.

Since $(5-4x)^{-7} = \frac{1}{5^7}\left(1-\frac{4}{5}x\right)^{-7}$.

we have $\frac{T_{r+1}}{T_r} = \frac{-7-r+1}{r}\left(-\frac{4}{5}x\right)$, *numerically*

$$= \frac{7+r-1}{r} \cdot \frac{4}{5} \cdot \frac{1}{2}$$

$$= \frac{2(6+r)}{5r}$$

Hence $T_{r+1} >, =,$ or $< T_r$

according as $\frac{2(6+r)}{5r} >, =,$ or < 1

i. e. according as $12+2r >, =,$ or $< 5r$

i. e. according as $12 >, =,$ or $< 3r$

i. e. according as $r <, =,$ or > 4.

Thus for all values of r, upto 3, T_{r+1} is greater than T_r ; when $r=4$ then $T_{r+1}=T_r$ and for greater values of r, T_{r+1} is less than T_r

Hence 4th and 5th terms are the greatest.

To find the value of the greatest term, we see that

$$T_{4+1} = \frac{(-7)(-7-1)(-7-2)(-7-3)}{4}\left(-\frac{4}{5}x\right)^4$$

$$= \frac{7.8.9.10}{2.3.4.} \times \frac{4^4}{5^4} \cdot \frac{1}{2^4}$$

$$= \frac{672}{125}.$$

Ex. 7. From what term will the terms of $(1-x)^{\frac{3}{2}}$ have the same sign, x being positive, and what is that sign ?

We know that $T_r+1 = T_r \cdot \frac{n-r+1}{r}(-n)$.

The two successive terms will have the same sign when $n-r+1$ begins to be negative *i. e.* when

$$\tfrac{3}{2}-r+1<0$$

i. e., when $r > 2\frac{1}{2}$

i. e., when $r=3$.

∴ the 3rd and 4th terms will have the same sign.

Now $T_3 = \frac{\frac{3}{2}(\frac{3}{2}-1)}{2}x^2$ = a positive quantity.

∴ The sign is positive.

Note It is obvious that all terms from and after the 3rd are of the same sign *i. e.*, positive.

EXERCISE XXVII.

1. Expand $(1-2x)^{-3}$ to 5 terms.

2. Expand $(1-x)^{\frac{1}{3}}$ to 4 terms.

3. Expand $\left(1+\frac{2x}{3}\right)^{\frac{1}{2}}$ to 4 terms.

4. Expand $(1-x^2)^{\frac{7}{3}}$ to 5 terms.

5. Expand $\left(1-\frac{2x}{a}\right)^{-\frac{1}{2}}$ to 4 terms.

6. Expand $(3a-2x)^{\frac{2}{3}}$ to 4 terms, and say for what values of x is the expansion valid.

7. Expand $\left(\frac{3}{a}-2x\right)^{-4}$ to 5 terms, and say for what values of x is the expansion valid ?

8. Expand $\frac{3a}{(a^3-x^2)^{\frac{1}{3}}}$ to 5 terms and give the general term.

9. What are the co-efficients of x^6 and x^{10} in $(a^3+3bx^2)^{-5}$?

10. Find the $(r+1)$th term in the expansion of $\frac{1}{\sqrt[4]{1-x}}$.

11. Find the general term of $(1-2x)^{-\frac{7}{2}}$.

12. Find the 14th term in the expansion of $(2^{10}-2^7.x)^{\frac{1}{2}}$.

13. Find the 3rd, 6th and rth terms in the expansion of $(a^2x^2-a^{-\frac{2}{5}}x^{\frac{9}{2}})^{-\frac{1}{2}}$. For what values of x is the expansion valid ?

14. Write down the rth term of $(a-x)^{-\frac{1}{n}}$.

15. What is the greatest term in $(1+x)^{\frac{3}{2}}$, when $x=\frac{5}{6}$?

16. Find numerically the greatest term in $(1+x)^{-n}$, when $x=\frac{1}{5}$ and $n=12$.

17. Which is the greatest term in $(1-7x)^{-\frac{11}{4}}$ when $x=\frac{1}{8}$?

18. Find the great st ter.n in $(1+x)^{\frac{2}{7}}$ when $x=\frac{2}{3}$.

19. Find the first negative term of $(1+x)^{\frac{13}{5}}$.

20. Give the first negative term of $\left(1+\frac{2x}{3}\right)^{\frac{12}{7}}$

21. What is the first negative term of $(7a+3x)^{\frac{14}{3}}$?

22. Which is the first negative term of $(5x+1)^{\frac{11}{5}}$?

23. Find the co-efficient of x^9 in $\frac{1+x}{(1-x)^2}$.

24. Find the co-efficient of x^5 in $\frac{(1+x)^2}{1-x}$.

25. Find the co-efficients x^r in the expansion of $\frac{1+x-x^2}{1+x}$.

26. What is the co-efficient of x^n in the expansion of $\frac{2+3x+4x^3}{(1-x)^4}$.

27. Find the $(r+1)$th term in the expansion of $(1+x+x^2+x^3+\ldots\ldots\text{to infinity})^n$.

28. Find the general term in $(1-3x+6x^2-10x^3+\ldots\ldots$ to infinity$)^n$, when $x<1$.

29. Show that $x^n=1+n\left(1-\frac{1}{n}\right)+\frac{n(n+1)}{2}\left(1-\frac{1}{x}\right)^2$ $+\ldots\ldots$to infin ty.

30. Show that $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}=1+\frac{1}{2}.\frac{x}{1+x}+\frac{1}{8}\left(\frac{x}{1+x}\right)^2$ $+\ldots\ldots$to infinity.

From which term will the terms of the following binomials have the same sign. Give the sign in each case.

31. $(1-x)^{\frac{10}{3}}$ **32.** $(1-x)^{\frac{7}{3}}$

33. $(1-x)^{10\frac{1}{4}}$ **34.** $(1-x)^{-3}$

75. Application of the Binomial Theorem. How the Binomial Theorem is applied will be clear from a few examples.

Ex. 1. If x be so small that its cube and higher powers may be neglected, prove that

$$\frac{(1-x)^{-\frac{5}{2}}+(16+8x)^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}+(2+x)^2}=1+\frac{23}{40}x^2. \qquad \text{(P. U. 1933)}$$

Since x is very small, we shall, in different expansions, retain terms upto x^2 only.

$$\frac{(1-x)^{-\frac{5}{2}}+(16+8x)^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}+(2+x)^2}=\frac{(1-x)^{-\frac{5}{2}}+4(1+\frac{x}{2})^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}+(2+x)^2}$$

$$=\frac{1+\frac{5}{2}x+\frac{35}{8}x^2+4(1+\frac{x}{4}-\frac{x^2}{32})}{1-\frac{1}{2}x+\frac{3}{8}x^2+4+4x+x^2}$$

$$=\frac{5+\frac{7}{2}x+\frac{17}{4}x^2}{5+\frac{7}{2}x+\frac{11}{8}x^2}$$

$$=(5+\tfrac{7}{2}x+\tfrac{17}{4}x^2)\times\tfrac{1}{5}(1+\tfrac{7}{10}x+\tfrac{11}{40}x^2)^{-1}$$

$$=(5+\tfrac{7}{2}x+\tfrac{17}{4}x^2)\times\tfrac{1}{5}\{1-(\tfrac{7}{10}x+\tfrac{11}{40}x^2)+(\tfrac{7}{10}x+\tfrac{11}{40}x^2)^2\}$$

$$=(5+\tfrac{7}{2}x+\tfrac{17}{4}x^2)\times\tfrac{1}{5}\{1-\tfrac{7}{10}x+\tfrac{43}{200}x^2\}$$

$$=\tfrac{1}{5}\{5+\tfrac{115}{40}x^2\}$$

$$=1+\tfrac{23}{40}x^2.$$

Ex. 2. Show that if n and N are nearly equal.

$$\left\{\frac{N}{n}\right\}^{\frac{1}{2}}=\frac{N}{N+n}+\frac{n+N}{4n} \text{ nearly.} \qquad \text{(P. U. 1925)}$$

Since N and n are nearly equal, let $N=x+\lambda$ and $n=x-\lambda$ so that λ is very small and its cube and higher powers may be neglected. Then

$$\left\{\frac{N}{n}\right\}^{\frac{1}{2}}=\left(\frac{x+\lambda}{x-\lambda}\right)^{\frac{1}{2}}=\frac{(x^2-\lambda^2)^{\frac{1}{2}}}{x-\lambda}=\frac{x}{x-\lambda}\left(1-\frac{\lambda^2}{x^2}\right)^{\frac{1}{2}}$$

$$=\frac{x}{x-\lambda}\left(1-\frac{\lambda^2}{2x^2}\right) \text{ nearly}$$

$$=\frac{2x^2-\lambda^2}{2x(x-\lambda)}=\frac{(x^2-\lambda^2)+x^2}{2x(x-\lambda)}$$

$$= \frac{x+\lambda}{2x} + \frac{2x}{4(x-\lambda)}$$

$$= \frac{N}{N+n} + \frac{N+n}{4n}.$$

Ex. 3. Evaluate $(3128)^{\frac{1}{5}}$ to 5 places of decimal.

$$(3128)^{\frac{1}{5}} = (3125+3)^{\frac{1}{5}} = (5^5+3)^{\frac{1}{5}} = 5\left[1+\frac{3}{5^5}\right]^{\frac{1}{5}}.$$

$$= 5\{1 + \frac{1}{5}\cdot\frac{3}{5^5} + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2}\cdot\frac{9}{5^{10}} + \ldots\}$$

$$= 5\{1 + \frac{3}{5^5} - 2\cdot\frac{9}{5^{12}} + \ldots\}$$

$$= 5\{1 + \frac{3.2^6}{10^6} - 2\cdot\frac{9\times 2^{12}}{10^{12}} + \ldots\}$$

$$= 5(1 + \frac{192}{10^6} - \frac{9.2^{13}}{10^{12}} + \ldots)$$

$$= 5\times 1.000192 = 5\ 00096, \text{ nearly.}$$

Ex. 4. Sum the series

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \ldots \text{to infinity.}$$

Let the given series be identical with

$$1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \text{to infinity whose sum is } (1+x)^n.$$

Comparing the second and third terms of the two series we have $nx = \frac{3}{4}$...(1) and $\frac{n(n-1)}{2}x^2 = \frac{3.5}{4.8}$...(2).

Divide (2) by the square of (1). Then

$$\frac{n(n-1)}{n^2} = \frac{15}{16}\times\frac{16}{9}$$

or $$1 - \frac{1}{n} = \frac{5}{3}$$

$$\therefore \quad n = -\tfrac{3}{2};$$

and hence from (1) $x = -\frac{1}{3}$.

$$\therefore \text{ the sum of the given series} = (1-\tfrac{1}{3})^{-\frac{3}{2}} = (\tfrac{2}{3})^{-\frac{3}{2}}$$

$$= 2^{\frac{3}{2}} = \sqrt{8}.$$

Ex. 5. Find the coefficient of x^{2n-1} in the expansion of $\frac{x+1}{(x-1)(1+x^2)}$, $x<1$.

Spliting $\frac{x+1}{(x-1)(1+x^2)}$ into partial fractions we get $\frac{x}{1+x^2}-\frac{1}{1-x}$.

Now $\frac{x}{1+x^2}=x(1+x^2)^{-1}$

$=x\{1-x^2+x^4-x^6+\ldots+(-1)^r x^{2r}+\ldots\}$

$=x-x^3+x^5-x^7+\ldots+(-1)^r x^{2r+1}+\ldots$

and $\frac{1}{1-x}=(1-x)^{-1}$

$=1+x+x^2+x^3+\ldots\ldots$

Hence the required coefficient = the coefficient of x^{2n-4} in $\frac{x}{1+x^2}$.

$=-$ the coefficient of x^{2n-1} in $\frac{1}{1-x}$,

$=(-1)^{r-1}-1$.

EXERCISE XXVIII

If x be so small that its square and higher powers may be neglected, find the approximate value of :—

1. $(1-2x)^{\frac{1}{2}}(1+5x)^{-\frac{2}{3}}$.

2. $\frac{(1-3x)^{\frac{1}{2}}+(1-x)^{\frac{5}{3}}}{(4-x)^{\frac{1}{2}}}$.

3. $\frac{(4+7x)^{\frac{2}{3}}}{(2+3x)(3-2x)^{\frac{1}{2}}}$.

4. $\frac{(1+3x)^{\frac{1}{3}}-(1+5x)^{-\frac{1}{5}}}{(1+\frac{x}{3})^{\frac{1}{5}}+(1-\frac{x}{5})^{-\frac{1}{3}}}$.

5. $\dfrac{(25+2x)^{\frac{1}{2}}-(1+\frac{3}{5}x)^{\frac{5}{3}}}{(16+3x)^{\frac{1}{2}}}$.

6. $\dfrac{(27-11x)^{\frac{1}{3}}-(1-3x)^{\frac{1}{7}}}{(1-4x)^{\frac{1}{2}}+(9+\frac{x}{3})^{\frac{1}{2}}}$.

7. Find the seventh root of 127 to five places of decimal.

8. Evaluate $(344)^{\frac{1}{5}}$ to four places of decimal.

9 Find the value of $\sqrt[3]{128}$ to five places of decimals.

10. What is the fifth root of 30 correct to five places of decimal ?

11. What is the value of $(630)^{-\frac{3}{4}}$ to five places of decimal.

12. Find to 4 places of decimal the value of $(1.3)^{\frac{3}{5}}$.

13. If x nearly equals unity, show that

$$ax^a-bx^b=(a-b)x^{a+b}.$$

14. If p and q are nearly equal, prove that

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\left(\frac{p}{q}\right)^{\frac{1}{n}}.$$

15. Sum $1+\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}+$.....to infinity.

16. Sum $1-\frac{1}{4}+\frac{1}{4}\cdot\frac{4}{8}-\frac{1}{4}\cdot\frac{4}{8}\cdot\frac{7}{12}+$.....to infinity.

17. Sum $1-\frac{15}{8}+\frac{15}{8}\cdot\frac{21}{16}-\frac{15}{8}\cdot\frac{21}{16}\cdot\frac{27}{24}+$.....to infinity.

18. Sum $1+\dfrac{1}{3}+\dfrac{1.3}{3.6}+\dfrac{1.3.5}{3.6.9}+\dfrac{1.3.5.7}{3.6.9.12}+$.....to infinity.

19. Sum $1+\dfrac{1}{3}+\dfrac{2.5}{6\ 12}+\dfrac{2.5.8}{6.12.18}+$.....to infinity.

20. Sum $\dfrac{1}{2}+\dfrac{1.3}{4.8}+\dfrac{1.3.5}{4.8.24}+$.....to infinity.

21. Sum $1+\dfrac{1}{10^2}+\dfrac{1.3}{1.2.10^4}+\dfrac{1.3.5}{1.2.3.10^6}+$.....to infinity.

22. Sum $1+\frac{1}{6}+\frac{1.4}{6.12}+\frac{1.4.7}{6\cdot12.18}+$.....to infinity.

23. Identify as a binomial expansion, the series and find its value

$$1+3.\frac{3}{16}+3^2.\frac{3.7}{16.32}+3^3.\frac{3.7.11}{16.32.48}+\text{.....to infinity.}$$

24. Show that

$$\left(\frac{1+2x}{1+x}\right)^n=1+n\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{2}\left(\frac{x}{1+2x}\right)^2+\text{.....}$$

to infinity.

25. Prove that $(\frac{9}{10})^{\frac{4}{5}}=$ ·9191662 nearly.

26. What is the coefficient of x^r in the expansion of $\frac{2+x}{x^2-1}$ when $x<1$?

27. Find the coefficient of x^r in the expansion of $\frac{1}{2x^2-x-1}$ when $x<\frac{1}{2}$.

28. Find the coefficient of x^{2n} in the expansion of $\frac{x^2-x+2}{1-x+x^2-x^3}$.

29. Prove that

$$\frac{1}{(1-x)(1-2x)(1-3x)}$$
$$=\tfrac{1}{2}(1-x)^{-1}-4(1-2x)^{-1}+4\tfrac{1}{2}(1-3x)^{-1}$$

and hence show that the coefficient of x^n in the given fraction is $\frac{1}{2}(3^{n+2}-2^{n+3}+1)$.

Miscellaneous Questions III.

1. Find the term independent of x in

$$\left(\frac{1}{3}x^2-\frac{3}{2x}\right)^9.$$

2 For what value of r will the coefficients of x^r and x^{n+1} in the expansion of $(1+x)^{2n+1}$ be equal ?

3. Show that

$$(1+x)^{-n}=\frac{1}{2^n}\{1+n\frac{1-x}{1+x}+\frac{n(n-1)}{2}\left(\frac{1-x}{1+x}\right)^2+\text{.....}$$

to infinity).

4. If x is very small show that

$$\frac{(1-3x)^{\frac{1}{3}}+3(1-x)^{\frac{5}{2}}}{(4-x)^{\frac{1}{2}}}=2-\frac{17}{4}x.$$

5. What binomial expansion will give the series

$1+\frac{7}{18}+\frac{7}{18}\cdot\frac{9}{36}+\frac{7}{18}\cdot\frac{9}{36}\cdot\frac{11}{54}+\ldots\ldots$ [P. U. 1930].

6. If p and q are positive integers show that the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ are equal.

7. What is the coefficient of x^8 in the expansion of $(1-3x)^{-\frac{5}{4}}$?

8. Show that the sum of the coefficients of odd terms in the expansion of $(1+x)^{2n}$ is 2^{2n-1}.

9. Show that $(1+2x+3x^2+4x^3+\ldots\ldots\text{to infinity})^{-1}=1-2x+x^2$

10. Find the value of $(244)^{\frac{1}{5}}$ correct to three decimal places.

11. Find the coefficient of x^4 in the expansion of $(1-x+x^2-x^3)^{-1}$

12. If M differs from N^2 by a very small quantity, then approximately

$$\sqrt{M}=\frac{3}{2}N-\frac{(3N^2-M)^2}{8N^3}.$$

13. Evaluale $\sqrt[3]{1010}$ to four places of decimal

14. Find the value of $(1{\cdot}1)^{\frac{3}{5}}$ to 3 decimal places.

15. Sum $1+\frac{1}{6}+\frac{1.3}{6.12}+\frac{1.3.5}{6.12\,18}+\ldots\ldots$to infinity.

16. Find the coefficient of x^6 in the expansion of $(1+3x+6x^2+10x^3+\ldots\ldots\text{to}\infty)^3$.

17. Find the coefficient of x^n in the expansion

$$\frac{1+x-2x^2}{(1-x)^3}.$$

18. Find a and b so that the coefficient of x^n in the expansion of $\frac{a+bx}{(1-x)^2}$ may be $4n+3$.

19. If x be very small, prove that

$$\frac{(1+2x)^{\frac{1}{5}}+(1-x)^{\frac{1}{3}}}{(1+2x)^{\frac{1}{4}}}=2-\tfrac{1}{1}\tfrac{4}{5}x.$$

20. Prove that $C_0-2C_1+3C_2-\ldots\ldots\ldots+(-1)^n(n+1)C_n=0.$

21. Find the co-efficient of x^4 in the expansion of $(1-2x+3x^2-4x^3+\ldots\ldots\ldots)^{-\frac{3}{2}}$.

22. Sum the series $1+\frac{1}{10}+\frac{1.4}{10.20}+\frac{1.4.7}{10.20.30}+\ldots\ldots\ldots\infty$

23. Find the co-efficient of x^r in the expansion of $\frac{9-21x}{(1-x)^3}$ and show that there is no term involving x^{19}.

24. Show that the co-efficient of x^r in the expansion of $(3-4x)(1-2x)^{-2}(1-x)^{-2}$ is $(r+1)(2^{r+2}-1)$

[Split up $\frac{3-4x}{(1-2x)^2(1-x)^2}$ into partial fractions and expand.]

25. Show that

$$1+1+\frac{3}{2.2}+\frac{3.4}{6.2^2}+\frac{3.4.5}{24.2^3}+\ldots\ldots\ldots$$
$$=1+\tfrac{3}{4}+(\tfrac{3}{4})^2+\ldots\ldots\ldots$$

26. Prove that $1+\frac{1}{2}+\frac{5}{2\,|\,2}\cdot\frac{1}{6}+\frac{5.7}{2\,|\,3}\cdot\frac{1}{6^2}+\frac{5.7.9}{2\,|\,4}\cdot\frac{1}{6^3}+\ldots\ldots$

$$=\sqrt{3}\left(1+\frac{1}{16}-\frac{1}{|\,2}\cdot\frac{1}{16^2}+\frac{3}{|\,3}\cdot\frac{1}{16^3}-\frac{3.5}{|\,4}\cdot\frac{1}{16^4}+\ldots.\right)$$

27. Prove that

$$1-n^2+\frac{n^2(n^2-1^2)}{1^2.2^2}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2}+\ldots=0,$$

where n is some positive integer.

[Use the expansions of $(1+x)^n$ and $(1+\frac{1}{x})^{-n}$, and find the term independent of x in the expansion of the product.]

28. If a, b, c, d be the co-efficients of any four consecutive terms in the expansion of $(1+x)^n$, show that $\frac{a}{a+b}, \frac{b}{b+c}, \frac{c}{c+d}$ are in A. P.

29. Show that the sum of the first r co-efficients in the expansion of $(1-x)^{-n}$ is $\frac{(n+1)(n+2)\ldots(n+r-1)}{\lfloor r-1}$.

[Use the expansions of $(1-x)^{-n}$ and $(1-x)^{-1}$ and find the coefficient of x^{r-1} in the expansion of the product.]

EXERCISE XXIX

Miscellaneous

1. Solve the equations

(*i*) $\frac{(a+b)(x+2)}{cx+a+b} = \frac{a+bx}{a+cx} + \frac{b+ax}{b+cx}$.

(*ii*) $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} + \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{17}{4}$; $x^2+y^2=706$.

2. In how many ways may 20 shillings be given to 5 persons so that no person may get less than 3 shillings ?

3. Solve (*i*) $\frac{\sqrt{x-5}}{3x-7} = \frac{\sqrt{3x-7}}{8(x-5)}$

(*ii*) $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{xyz}{x+y+z}$.

[Put each fraction equal to k.]

4. Show that if bc, ca, ab be in H. P., then

$$a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$$

will be in A. P.

5. If a, b, c be in G. P. and if p be the A. M. between a and b and q the A. M. between b and c, then b will be the H. M. between p and q.

6. Show that the ratio between the nth and $(n-1)$th co-efficients in the expansion of $(1-x)^{-n}$ is 2 : 1, where n is a positive integer.

7. Show that $ax^2+bx+c=0$ is an identity when it is satisfied by α, β, Υ, all different values of x.

Find if $9(x-1)(x-2)+(x-3)(x-2)-8(x-1)(x-3)=2x^2$ is an identity.

8. For all real values of x find the greatest value of $\frac{2+x}{6+3x+2x^2}$.

9. What is the co-efficient of x^n in $\frac{(1+3x)^3}{(1+2x)^2}$?

10. Find the general term of $(1+x)^{-n}$ and the middle term of $(x+\frac{1}{x})^{2n}$, when n is a positive integer.

11. Show that the roots of $x^2(b^2+d^2)+2x(ab+cd)+a^2+c^2=0$ will be equal if they are real.

12. Find the cube root of 126 to five places of decimal.

13. Find x from $\sqrt[m]{(a+x)^2}+2\sqrt[m]{(a-x)^2}=3\sqrt[m]{(a^2-x^2)}$.

14. Find 4 numbers in A. P. which when increased by 1, 1, 3, 9 respectively give a G. P.

15. Given a, b, c in A. P. and a^2, b^2, c^2 in H.P., prove that $-\frac{a}{2}, b, c$ are in G. P. or $a=b=c$.

16. Solve the equations (*i*) $\frac{a^2}{(x-a)^2}=\frac{b^2}{(x+b)^2}$.

(*ii*) $\frac{x^2+y^2}{xy}+x^2+y^2=\frac{40}{3}$.

$\frac{xy}{x^2+y^2}+xy=\frac{33}{10}$.

[Put $y=vx$ and divide.]

17. Find the co-efficient of x^r in $\frac{1+x-2x^2}{(1-x)^2}$.

18. Show that $1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\frac{1.4.7.10}{4.8.12.16}+\cdots$

$=1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\frac{2.5.8.11}{6.12.18.24}+\cdots\cdots$

19. How many different sums of money can be represented with a soverign, a half-soverign, a crown, a florin, a shilling, a penny and a farthing ?

20. Solve (*i*) $\frac{6}{x^2-1}+\frac{x}{x+1}=\frac{5}{2}$

(*ii*) $3xy+2z=xz+6y=2yz+3x=0$.

[Transpose each equation and multiply together.]

21. Solve

(*i*) $\sqrt{x^2+a}+\sqrt{x^2-a}=\sqrt{2a+b}+\sqrt{b}$.

(*ii*) $x(3y-5)=4$; $y(2x+7)=27$.

22. Two students attempt the same quadratic equation. The first after reducing has only a mistake in the coefficient of x and gets the roots $-9, -1$; while the second after reducing has only a mistake in the numerical term and gets roots 8, 2. Find the roots of the correct equation.

23. An A. P., *a* G. P, and an H. P. have $\frac{1}{3}$ and $-\frac{1}{4}$ as their first two terms, find their third terms.

24. Find n from (*i*) $\frac{1}{7}\frac{1}{2}\,{}^{n}P_5={}^{n}P_3$.

(*ii*) ${}^{20}C_{2n-3}={}^{20}C_{3n-2}$.

25. In how many ways can an eleven be selected from 25 players with the condition that 6 of them are always included and five always excluded ?

26. How many Words framed with the letters of the word *equation* will begin and end with a consonant ?

27. Show that

$$\left[\frac{a+x}{a-x}\right]^{\frac{1}{2}}=1+\frac{x}{a+x}+\frac{3}{2}\left[\frac{x}{a+x}\right]^2+\frac{5}{2}\left[\frac{x}{a+x}\right]^3+\ldots\ldots$$

28. Out of n things only two are alike ; show that the number of combinations of them taken r at a time is ${}^{n}C_r-{}^{n-2}C_{r-1}$.

29. In how many ways can 24 tables and 23 chairs be arranged in a line so that no two chairs may be together; all tables and all chairs being alike.

30. If $x<1$, prove that

$$\frac{x}{1-x}-\frac{x^3}{1-x^3}+\frac{x^5}{1-x^5}-\ldots\ldots\text{to infinity}$$

$$=\frac{x}{1+x^2}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x^6}+\ldots\ldots\text{to infinity.}$$

[$\frac{x}{1-x}$ is the sum of the infinite G. P. $x+x^2+x^3+\ldots$; $\frac{x^3}{1-x}$

is the sum of the infinite G. P. $x^3+x^6+x^9+\ldots$; and so on

$\therefore$ the first series

$$=(x+x^2+x^3+\ldots\ldots)-(x^3+x^6+x^9+\ldots\ldots)+$$
$$(x^5+x^{10}+x^{15}+\ldots\ldots)-\ldots\ldots\ldots$$
$$=(x-x^3+x^5-\ldots\ldots)+(x^2-x^6+x^{15}-\ldots\ldots)+$$
$$(x^3-x^9+x^{15}-\ldots\ldots)+\ldots\ldots\ldots$$
$$=\frac{x}{1+x^2}+\frac{x^2}{1+x^4}+\frac{x^3}{1+x^6}+\ldots\ldots\ldots\ldots\Big].$$

31. If S_1, S_2, S_3, be the sums of n, $2n$, $3n$ terms respectively of a G. P, prove that $S_1(S_3-S_2)=(S_2-S_1)^2$.

32. Prove that $n^2+2n\{1+2+3+\ldots\ldots+(n-1)\}=n^3$.

33. Find r if the coefficients of the rth and $(r+4)$th terms in the expansion of $(1+x)^{20}$ are equal.

34. For real values of x show that the maximum value of $2(a-x)(x+\sqrt{x^2+b^2})$ is a^2+b^2.

35. Sum the infinite series

$$1-\frac{1}{5}-\frac{1}{5^2}-\frac{2}{6.5^2}-\ldots\ldots\ldots$$

36. Solve

$$(i)\quad \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}+\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}=98.$$

$$(ii)\quad x^2+2xy-y^2=ax+by$$
$$x^2-2xy-y^2=bx-ay.$$

$$\left(\text{Put } y=vx \text{ and divide. Thus get } v=\frac{a-b}{a+b}\right).$$

37. If m is the number of combinations of n things taken 2 together, then ${}^{m}C_2=3\,{}^{n+1}C_4$.

38. If p, q, r be inA. P. and if $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$, show that x, y, z are in H. P.

$$\left(\text{The given equations are } \frac{\frac{a}{x}-1}{p}=\frac{\frac{a}{y}-1}{q}=\frac{\frac{a}{z}-1}{r}\right.$$

$$\therefore\quad \frac{a\left[\frac{1}{x}+\frac{1}{z}\right]-2}{p+r}=\frac{2\left[\frac{a}{y}-1\right]}{2q}$$

$$\therefore \quad \frac{1}{x}+\frac{1}{z}=\frac{2}{y}\Big)$$

39. A sum of money is distributed amongst a certain number of men. The second man receives 1s. more than the first, the third 2s. more than the second, the forth 3s. more than the third. and so on. If the first man receives 1s. and the last man £3. 7s., find the number of men and the sum distributed.

40. A gentleman has got 5 sorts of note-paper, 8 different pens and 4 different inks ; in how many ways can he begin to write a letter ?

41. Show that $\dfrac{(1+2x)^{\frac{1}{6}}+(1-x)^{\frac{1}{3}}}{(1+2x)^{\frac{1}{4}}}=2-\dfrac{14}{15}x$, when x is very small.

42. There are chalk sticks of 7 prismatic colours. In how many ways can they be arranged so that the violet and red sticks shall never be together ?

43. Solve $\sqrt{3x-2}-\sqrt{2x-1}=\sqrt{5x-4}-\sqrt{4x-3}$.

44. Find the value of $(1.02)^{\frac{1}{3}}$ to four places of decimal.

45. Find the value of $\dfrac{(4+\sqrt{15})^{\frac{3}{2}}+(4-\sqrt{15})^{\frac{3}{2}}}{(6+\sqrt{35})^{\frac{3}{2}}-(6-\sqrt{35})^{\frac{3}{2}}}$.

$[(4+\sqrt{15})^{\frac{3}{2}}=(4+\sqrt{15})\sqrt{4+\sqrt{15}}=(4+\sqrt{15})\times(\sqrt{\tfrac{5}{2}}+\sqrt{\tfrac{3}{2}})=$etc.]

46. Prove that $w^4+2w^6+w^8=1$ where w and w^2 are the imaginary cube roots of unity.

47. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it is either

$$\frac{q-q'}{p-p'} \text{ or } \frac{pq'-p'q}{q-q'}.$$

48. In an A. P., whose first term is a, if the sum of the first p terms is zero, show that the sum of the next q terms is $-\dfrac{a(p+q)q}{p-1}$.

49. If one of the roots of $x^2+px+q=0$ is the square of the other, prove that $p^3-q(3p-1)+q^2=0$.

50. If α, β are the roots of $x^2+px+q=0$ find the values of

(i) $\alpha^2+\alpha\beta+\beta^2$, (ii) $\alpha^3+\beta^3$, (iii) $\alpha^4+\alpha^2\beta^2+\beta^4$.

PUNJAB UNIVERSITY PAPERS.

1925

1. Solve for x, $\frac{x-a}{x+a}+\frac{x+a}{x-a}=\frac{34}{15}$.

If one root of the equation $x^2-px+q=0$ be the square of the other, find the relation between p and q.

2. Sum to n terms the following series :—

(*i*) $(n-1)+(n-2)+(n-3)+\ldots\ldots\ldots$

(*ii*) $a+(a+d)r+(a+2d)r^2+\ldots\ldots\ldots$

(*iii*) $1+3+7+15+31+\ldots\ldots\ldots$

3. (*a*) Obtain the number of combinations of n things taken r at a time. In how many of these will three given things not occur ?

(*b*) Show that if n and N are nearly equal,

$$\left[\begin{matrix}\mathrm{N}\\ n\end{matrix}\right]^{\frac{1}{2}}=\frac{\mathrm{N}}{\mathrm{N}+n}+\frac{n+\mathrm{N}}{4n} \text{ nearly.}$$

Evaluate $\sqrt[3]{995}$ to 4 places of decimal by the Binomial Theorem.

1926

1. (*i*) Solve $a(x^2+1)=x(a^2+4)$.

(*ii*) If α and β are the roots of $x^2-3x+1=0$, find the equation whose roots are $\frac{\alpha}{1+\beta}$, $\frac{\beta}{1+\alpha}$.

Or

Euler says it would be difficult to find x so as to make $p^4-2p^2x^2+x^4+x$ a square, but show that it is not difficult.

2. Sum the series :—

(*i*) $51+50+49+\ldots\ldots\ldots+21.$

(*ii*) $1+2r+3r^2+4r^3+\ldots\ldots\ldots+nr^{n-1}.$

(*iii*) $x+2(x-\frac{1}{n-1})+3(x-\frac{2}{n-1})+4(x-\frac{3}{n-1}$
$+\ldots\ldots$to n terms.

3. (*i*) Find the number of combinations of n things r at a time.

(*ii*) Evaluate $\{1+\frac{3}{10}\}^{\frac{8}{5}}$ to 4 places of decimal by the Binomial Theorem.

(*iii*) If N and n be nearly equal, then

$$\sqrt{\frac{N}{n}} = \frac{N}{N+n} + \frac{1}{4}\frac{N+n}{n} \text{ nearly.}$$

1927

1. Solve the equation $ax^2+bx+c=0$.

If α and β be its roots, show that for all values of x

$$ax^2+bx+c=a(x-\alpha)(x-\beta),$$

and find the equation whose roots are $\alpha+\frac{1}{\alpha}$, $\beta+\frac{1}{\beta}$.

2. Sum up the series $1^3+2^3+3^3+\ldots\ldots+n^3$.

If m is the Arithmetic mean of n consecutive integers, show that the sum of their cubes is

$$mn\{m^2+\tfrac{1}{4}(n^2-1)\}.$$

3. From the first principles prove that

$${}^nC_r=\frac{n}{r}\,{}^{n-1}C_{r-1}$$

Show that ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$.

4. Prove the Binomial Theorem for a positive integral index and find to four decimal places the fifth root of 344.

1928

1. Find the condition that the roots of $ax^2+bx+c=0$ may be (*i*) equal (*ii*) real and unequal.

Show that the roots of

$$(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$$

are real When will the roots of this equation be equal ?

2. Show that the sum to infinity, of the series $a+ar+ar^2+\ldots\ldots=\frac{a}{1-r}$ where the numerical value of r is less than unity.

Evaluate : $1+4x+9x^2+16x^3+25x^4+\ldots\ldots$ to infinity, when $x=\frac{1}{5}$.

3. Find the number of permutations of n dissimilar things taken r at a time.

Eight papers are set in an examination, two of them in Mathematics. In how many differeut orders can the papers be given, provided that the two Mathematics papers are not successive ?

4. State (without proof) the Binomial Theorem for negative or fractional index.

Expand $(1+x+x^2)^{\frac{1}{2}}$ in ascending powers of x upto the terms containg x^3. for what values of x is the expansion valid ?

1929

1. Solve the equation $x^2+px+\ =0$, and show that the necessary and sufficient condition that its roots should be rational is that $p=k+\frac{q}{k}$, where p, q, k are rational.

2. If the mth term of a series in H. P. be n, and the nth term be m, prove that the rth term is $\frac{mn}{r}$.

Prove that ${}^{n}+{}^{1}C_r={}^{n}C_r+{}^{n}C_{r-1}$.

3. Sum the series :—

(*i*) $1^2+2^2+3^2+4^2+$ .....to $2n$ terms.

(*ii*) $1-x+2x^2-3x^3+4x^4-$ .....to infinity $(0<x<1)$.

4. Write down the expansion of $(1-x)^{-n}$ when $0<x<1$, and identify the series

$$1+\tfrac{1}{8}+\tfrac{1}{8}\cdot\tfrac{2}{16}+\tfrac{1}{8}\cdot\tfrac{2}{16}\cdot\tfrac{3}{24}+\ldots\ldots\ldots$$

as a binomial expansion.

Show that $\left[\frac{9}{10}\right]^4=\cdot 9191662$ nearly.

1930

1. Prove that an equation of the 2nd degree in x cannot have more than two roots, and solve the equation

$$\frac{1}{x+1}=\frac{1}{x+2}.$$

Between what limits does $\frac{x^2-2x+9}{x^2+2x+9}$ lie (x being real) ?

2. Find the sum of the squares of the first n odd integers.

If n is even and $\alpha+\beta$ and $\alpha-\beta$ are the middle pair of terms, show that the sum of the cubes of n terms in A. P. is $n\alpha\{\alpha^2+(n^2-1)\beta^2\}$.

3. Find the number of combinations of n things taken r at a time:

If ${}^{n+2}C_8 : {}^{n-2}C_4 = 57 : 16$, find n.

4. (*i*) Sum the series $1+x+x^2+\ldots\ldots+x^n$.

If the number of terms be infinite, determine when the sum is finite and definite.

(*ii*) State the Binomial Theorem and indentify the series

$$1+\tfrac{7}{18}+\tfrac{7}{18}\cdot\tfrac{9}{36}\cdot+\tfrac{7}{18}\cdot\tfrac{9}{36}\cdot\tfrac{11}{54}+\ldots\ldots\ldots$$

as a binomial expansion.

1931

1. (*i*) Solve the equation $ax^2+bx+c=0$, and obtain the condition that its roots may be negative (a, b, c, being all real).

(*ii*) If α, α^2 be its roots, find c when $a=8$ and $b=-30$.

2. (*i*) Find the binomical expansion of which four consecutive terms are 576, 2160, 4320, 4860.

(*ii*) Prove that the coefficient of x^n in the expansion of $\left[\frac{1+x}{1-x}\right]^2$ is $4n$; $(x<1)$. Point out an exceptional case, if any.

3. (*i*) Prove without assuming the formula for the number of permutations that

$${}^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}.$$

(*ii*) Solve the equations :—

$$xy(x+y)=30,\quad x^3+y^3=35.$$

4. (*i*) Prove $\log_b{}^n = \log^{a^n} \times \log_b{}^a$.

(*ii*) If a, b, c are in G. P., show that $\log_a{}^n$, $\log_b{}^n$, $\log_c{}^n$ are in H. P.

5. Sum the series :—

(*i*) $1^2-2^2+3^2-4^2+\ldots+(2n-1)^2-(2n)^2$.

(*ii*) $1+3x+4x^2+5x^3+\ldots$to ∞ $(x<1)$.

1932

1. (*i*) Solve the equation $ax^2+bx+c=0$.

(*ii*) Find the number of terms of an A. P. of which the first term, the sum, and the common difference are $\frac{5}{7}$, 5, $-\frac{1}{2}\frac{1}{1}$ respectively.

2. (*i*) Sum the series $a+ar+ar^2+\ldots+ar^n$.

(*ii*) Find the least value of n for which $S-S_n<03$, where S denotes the sum to infinity and S_n the sum of n terms of the series

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{1}\tfrac{1}{6}+\ldots$$

3. Prove from the first principles that ${}^nC_r={}^{n-1}C_{r-1}$ Apply it to show that ${}^nC_o+{}^nC_1+{}^nC_2+\ldots+{}^nC_n=2^n$.

4. (a) State the Binomial Theorem and apply it to find the value of (*i*) $(999)^4$, (*ii*) $(129)^{\frac{1}{7}}$ to three decimal places.

(b) Sum the series $1-x+2x^2-3x^3+4x^4-\ldots\infty$.

1933

1. Discuss the nature of the roots of the equation $ax^2+2bx+c=0$.

If α, β be the roots of the equation $x^2+x+1=0$, form the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$ and account for the identity of the equation thus obtained with the original equation.

2. Sum the series :—(*i*) $1^3+2^3+3^3+\ldots+n^3$.

(*ii*) $2+2+\frac{8}{3}+1+\frac{5}{8}+\ldots\infty$.

If S denotes the sum, and S_n the sum of the first n terms of the series (*ii*), find n for which $S-S_n<$ or $=\frac{1}{2}$.

3. Expand $(1+x)^m$ in powers of x when m is a positive integer.

If x be so small that its cube and higher powers may be neglected, prove that

$$\frac{(1-x)^{-\frac{5}{2}}+(16+8x)^{\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}+(2+x)^2}=1+\frac{23}{40}x^2.$$

4. (*a*) Define permutations, and prove from the first principles that ${}^{n}P_{r}={}^{n-1}P_{r}+r.{}^{n-1}P_{r-1}$.

(*b*) Find four numbers in G. P. so that their sum may be 15 and the sum of their squares 85.

Or

(*c*) Prove that if the equation in x, $(ay+\lambda)x^2+2(by+u)x+(cy+v)=0$ has a root independent of y, then both roots are rational (a, b, c, λ, u, v, y being rational numbers).

1934

1. (*a*) If the equations $x^2+px+q=0$ and $x^2+rx+s=0$ have a common root, prove that it is either $\frac{ps-rq}{q-s}$ or $\frac{q-s}{r-p}$.

(*b*) Solve the equations (*i*) $x+\frac{1}{x}=3\frac{1}{3}$,

(*ii*) $\frac{x+2}{x-1}+2\frac{2}{3}=\frac{x+3}{x-2}$.

2. (*a*) Sum the following series to n terms :—

(*i*) $2.5+5.8+8.11+\ldots$

(*ii*) $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots$

(*b*) Express $.2\dot{3}\dot{4}$ as an infinite Geometrical series. Find the sum of the series in the form of a vulgar fraction and deduce the Arithmetical Rule for converting a mixed recurring decimal into a vulgar fraction.

3. (*a*) Find the number of combinations of n things taken r at a time. Prove that ${}^{10}C_4+{}^{10}C_3={}^{11}C_4$.

(*b*) How many different words can be formed out of the letters of the word *Amritsar* taken all together ?

4. (*a*) State and prove the Binomial Theorem for a positive integral index.

(*b*) Obtain to three places of decimal the fifth root of 31 (by using the Binomial Theorem).

(*c*) If x be very small so that its square and

higher powers may be neglected, prove that $\frac{1+x}{1-x}=1+2x$.

Or

(*c*) Split up into partial fractions the function $\frac{f(x)}{(x-a)(x-b)(x-c)}$.

1935

1. (*a*) If α, β be the roots of the equation $x^2+ax+\frac{1}{4}(a^2-b^2)=0$, prove that $\alpha+\beta, \alpha-\beta$ are the roots of the equatian $x^2+(a\pm b)x\pm ab=0$.

(*b*) Solve (*i*) $\frac{4}{x+4}-\frac{3}{x+3}=\frac{2}{x+2}-\frac{1}{x+1}$

(*ii*) $x^2+xy=15$, $x^2-y^2=5$.

2 (*a*) If *a*, *b*, *c* be in A. P. and *x*, *y*, *z* in G. P. prove that $x^by^cz^a=x^cy^az^b$.

(*b*) Sum the series (*i*) $1^2+2^2+3^2+\ldots\ldots+n^2$

(*ii*) $(1\frac{1}{3})^2+(2\frac{1}{3})^2+(3\frac{1}{3})^2+\ldots\ldots$ to *n* terms.

3. (*a*) Find the number of permutations of *n* dissimilar things taken *r* at a time.

Find the value of *n*, if ${}^nP_4 : {}^{n-1}P_3=9:1$.

(*b*) Find the number of ways in which *m* boys and *n* girls can be arranged in a row, so that no two of the girls are together. ($m>n$.)

4. (*a*) Apply Binomial Theorem to expand $(1-x)^{-n}$; and find the sum of the first *r* co-efficients of the expansion

(*b*) Split up into partial fractions the function $\frac{1}{(x+1)^2(x^2+1)}$.

1936

1. (*a*) Find the roots of the equation $ax^2+2bx+c=0$, and establish tests to determine the nature of these roots.

(*b*) Solve the equations (*i*) $x^{-2}-10=3x^{-1}$

(*ii*) $x+\frac{3}{y}=18,$

$y+\frac{6}{x}=1.$

2. (*a*) Find the sum of a number of terms in Geometrical Progression.
Sum the series $a+b$, a^2+2b, a^3+3b,...to n terms.

(*b*) If between any two quantities there be inserted two arithmetic means A_1, A_2; two geometric means G_1, G_2; and two harmonic means H_1, H_2; show that $G_1 G_2 : H_1 H_2 = A_1+A_2 : H_1+H_2$.

3. (*a*) Find the number of selections of n things taken r at a time.
If ${}^{n+1}C_{r+1} : {}^{n}C_{r} : {}^{n-1}C_{r-1}=11 : 6 : 3$, find n and r.

(*b*) Find the number of diagonals that can be drawn by joining the angular points of a seven-sided figure.

4. (*a*) When n is a positive integral number, write down the expansion of $(1+x)^n$, and find out the greatest term in the expansion

(*b*) Also find the sum of the squares of all the co-efficients in the same expansion.

Or (*b*) Split up into partial fractions,

$$\frac{x-4}{(x^2+4)(x^2-3x+2)}.$$

1937

1. (*a*) Solve the equation $ax^2+bx+c=0$, find the condition that the equations $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$ may have a common root.

(*b*) Solve the equations (*i*) $\sqrt{3x^2+1}+\frac{4}{\sqrt{3x^2+1}}=5$

(*ii*) $3x+4y=25$, $\frac{3}{x}+\frac{4}{y}=2.$

2. (*a*) Find the sum of n terms of a Geometrical Progression of which the first term is a and the second term is ar.

Sum the series $50-40+32-25.6+20.48+$ upto infinity.

(*b*) Between what values must x lie in order that the series $\frac{2x}{x+3}+\left[\frac{2x}{x+3}\right]^2+\left[\frac{2x}{x+3}\right]^3$ $+$...to infinity may have a definite sum ?

3. (*a*) Find the number of permutations of n different things taken r at a time.

(*b*) In how many ways can 5 rupees and 6 pice be arranged (*i*) in a row, (*ii*) in a ring (the rupees being all alike and the pice all alike) ?

4. (*a*) State and prove the Binomial Theorem for positive integral indices.

(*b*) Find the co-efficients of the two middle terms in $(1+x)^{2n+1}$, and the co-efficient of the middle term in $(1+x)^{2n}$, and show that the quotient of the first expression by the second always lies between $\frac{1}{2}$ and 2.

1938

1. Solve the equation $x^2+px+q=0$, and find the condition that one root of it may be n times the other.

If the difference of the corresponding roots of the equations $x^2+px+q=0$ and $x^2+qx+p=0$ be the same, then $p+q+4=0$

Solve the equation

$$(b-c)x^2+(c-a)x+(a-b)=0.$$

2. Sum the series

(*i*) $1^3+2^3+3^3+\ldots\ldots\ldots+n^3$.

(*ii*) $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots\ldots$to n terms.

If a^2, b^2, c^2 are in A. P., Show that $b+c$, $c+a$, $a+b$ are in H. P.

3. From first principles or otherwise prove that

$${}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$$

If ${}^nC_{12}={}^nC_8$, find ${}^{22}C_n$.

From 7 Unionists and 4 Congressites a committee of 6 is to be formed. In how many ways can it be done when the committee contains at least two Congressites ?

4. State the Binomial Theorem for any index and express the series

$$1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2^2}-\frac{1.3.5}{2.4.6}\frac{1}{2^3}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{1}{2^4}-\cdots$$

as a binomial expansion and find its value.

(*b*) If the successive coefficients in the expansion of $(1+x)^n$ be 220, 495 and 792, find n.

Or

Split up into partial fractions the following

$$\frac{2x+1}{(x+2)(x-3)^2}.$$

ANSWERS.

$$7^{-x} + 7^{1+x} - 8 = 0$$

$$\frac{1}{7^x} + 7 \cdot 7^x - 8 = 0$$

$$\frac{1}{7^x} + 7 \cdot 7^x - 8 = 0$$

$$\frac{1}{7^x} + 7 \cdot 7^x - 8 = 0$$

$$\frac{1}{7^x} + 7 \cdot 7^x - 8 = 0$$

$$\frac{1}{7^x} + 7 \cdot 7^x - 8 = 0$$

$$1 + 7^{2x} = 8 \cdot 7^x$$

ANSWERS.

Exercise I

1. $\frac{1}{3}$, 5. **2.** 5, 7. **3.** -7, 4. **4.** $\frac{2}{3}$, $-\frac{3}{2}$.
5. $-\frac{2}{7}$, $\frac{5}{7}$. **6.** 15, -11. **7.** $-\frac{7}{8}$, 3. **8.** 2, $\frac{1}{3}$.
9. $a+b$, $\frac{a+b}{2}$. **10.** $p+1$, $p-1$. **11.** $7a$, $3a$.
12. $\frac{a+b}{c}$, $\frac{b+c}{a}$. **13.** 17, 3. **14.** $1 \pm \sqrt{5}$.
15. $a \pm \sqrt{2}$. **16.** c, $\frac{1}{c}$. **17.** -1, -2.
18. 3, $\frac{1}{3}$. **19.** 6, -9. **20.** $\frac{4}{3}p$, $\frac{2}{3}p$.
21. $\frac{13}{4}$, $\frac{1}{2}$. **22.** $3 \pm \sqrt{5}$. **23.** $m \pm n$.
24. -9, 4. **25.** $\frac{m \pm n}{l}$. **26.** $1 \pm 2\sqrt{3}$.
27. $1 \pm \frac{\sqrt{3}}{2}$. **28.** $\frac{a \pm b}{c}$. **29.** $\pm \sqrt{\frac{2}{3}}$.
30. $\pm \frac{\sqrt{3}}{2}$. **31.** 0, $a+b$. **32.** $\pm \frac{a}{\sqrt{3}}$.
33. 1 sec., $1\frac{1}{2}$ sec. **34.** $-3 \pm 3\sqrt{5}$ in. **35.** 5, -4.

Exercise II

6. $36+i77$. **7.** $0+i(a^2+b^2)$. **8.** $1\frac{2}{3}+i1\frac{5}{9}$.
9. $\frac{1}{2}+i\frac{1}{2}$. **10.** $\frac{1}{2}\frac{7}{5}+i\frac{1}{2}\frac{9}{5}$.
11. $0+i\frac{4pq}{p^2+q^2}$. **27.** $\pm(2+i3)$. **28.** $\pm(5-i4)$.
29. $\pm\frac{1}{\sqrt{2}}(1+i)$. **30.** $\pm\sqrt{2}(1-i)$.

Exercise III

1. ± 5, $\pm\sqrt{3}$. **2.** $\pm\frac{1}{\sqrt{3}}$, $\pm i\frac{1}{2}$.
3. 2, $2w$, $2w^2$, $\frac{1}{2}$, $\frac{1}{2}w$, $\frac{1}{2}w^2$. **4.** 4, 9.
5. -1, 27. **6.** 81, 16.
7. $(-4a)^n$, $(3a)^n$. **8.** 0, 2. **9.** 2, 3.
10. 0, 2. **11.** 0, -1. **12.** $\frac{2}{3}$, $\frac{1}{3}$.
13. $1^{\frac{9}{9}}$, $1^{\frac{4}{9}}$. **14.** $\pm\frac{1}{\sqrt{2}}, \pm\sqrt{\frac{2}{5}}$.

15. $1, \frac{2}{3}$. **16.** $0, 4$.

17. $\frac{1}{2}(-5 \pm \sqrt{21}), \frac{1}{2}(-5 \pm i\sqrt{11})$

18. $3, -2, 8, -7$. **19.** $3, -\frac{3}{2}, \frac{1}{4}(3 \pm i\sqrt{47})$

20. $4, -\frac{7}{2}, \frac{1}{4}(1 \pm \sqrt{65})$. **21.** $-1, 4, \frac{3}{2}(1 \pm \sqrt{5})$.

22. 1. **23.** $7, -\frac{14}{3}$. **24.** $2, -\frac{1}{2}$.

25. $1, 2, \frac{1}{2}(3 \pm i)$. **26.** $\pm 5, \pm 3\sqrt{2}$. **27.** $2, -4\frac{1}{3}$.

28. $4, -2\frac{4}{5}$. **29.** $6, -\frac{5}{2}$. **30.** $3, 2$.

31. $1, \frac{1}{2}(-3 \pm \sqrt{5})$. **32.** $2 \pm \sqrt{3}, \frac{1}{2}(-1 \pm i\sqrt{3})$

33. $\frac{3 \pm i\sqrt{7}}{4}, \frac{-1 \pm i\sqrt{15}}{4}$. **34.** $1, \frac{3 \pm \sqrt{5}}{2}$.

35. $2, -\frac{1}{2}, 3, -\frac{1}{3}$. **36.** $3, \frac{1}{3}, \frac{-5 \pm i\sqrt{11}}{6}$.

37. $5, \frac{1}{2}$. **38.** $3, 5\frac{2}{3}$. **39.** $\frac{-5 \pm \sqrt{17}}{4}$.

40. $\frac{-6 \pm 2\sqrt{3}}{9}$. **41.** $-\frac{7}{5}, -\frac{2}{7}$. **42.** $\frac{-22 \pm 2\sqrt{37}}{3}$.

43. $2, \frac{2}{5}$. **44.** $\pm\frac{2a}{5}$. **45.** $3, -1$.

46. ± 1. **47.** $0, a$. **48.** $\frac{a}{2}(1 \pm \sqrt{5})$.

49. $a, -b$. **50.** $\pm\sqrt{\frac{n}{n-2}}, \pm\sqrt{\frac{n-1}{n+1}}$.

Exercise IV

1. (*i*) $x^2 - 7x + 12 = 0$ (*ii*) $x^2 + 2x - 24 = 0$
(*iii*) $x^2 + 4x - 45 = 0$ (*iv*) $3x^2 - 56x + 36 = 0$
(*v*) $7x^2 + 54x + 35 = 0$ (*vi*) $x^2 - 6x + 7 = 0$
(*vii*) $x^2 - 8x + 9 = 0$ (*viii*) $x^2 - 4x + 9 = 0$
(*ix*) $x^2 + 12x + 45 = 0$
(*x*) $(a^2 - b^2)x^2 + 4abx - a^2 + b^2 = 0$.

2. (*i*) $\pm i\frac{2}{3}\sqrt{11}$ (*ii*) $-\frac{14}{15}$ (*iii*) $\frac{116}{27}$.

3. (*i*) $\frac{m^2 - 2nl}{l^2}$ (*ii*) $\frac{m^2 - 2nl}{n^2}$ (*iii*) $-\frac{mn}{l^2}$.

4. (*i*) $4\frac{048}{81}$ (*ii*) $\pm\frac{5}{9}\sqrt{109}$ (*iii*) $8\frac{05}{9}$.

5. (*i*) $pq(p^2 - 3q)$ (*ii*) $\frac{p^2 + p - 2q}{1 + p + q}$ (*iii*) $\frac{2(1 - q)}{1 + p + q}$.

6. $2q + 2m - lp$. **8.** $qx^2 + 2px + 4 = 0$.

9. $mx^2 - (l^2 - 2m)x + m = 0$. **10.** $x^2 - a^2x + a^2b = 0$.

11. (*i*) $a^{11}x^2+a^4bc^4(b^2-3ac)x+c^{11}=0$
(*ii*) $a^2cx^2-b(3ac-b^2)x+ac^2=0$
(*iii*) $a^2c^2x^2-(b^4-4ab^2c+2a^2c^2)x+a^2c^2=0$.
12. $(a^2r+abq+b^2p)x^2-(aq+2bp)x+p=0$.
13. $x^2-x+3=0$. **14.** $x^2-2bx+4c=0$.
15. $x^2+(2x-m)x+n^2=0$. **16.** $ax^2+bkx+ck^2=0$.
17. $2m^2=9ln$. **18.** $4b^2=25ac$. **20.** 79.
21. $\begin{matrix}np-lr\\lq-mp\end{matrix}$ or $\begin{matrix}mr-nq\\np-lr\end{matrix}$. **23** -15, 0. **24.** 24, $19\frac{4}{9}$
25. 0 or 8.

Exercise V

1. (*i*) Imaginary (*ii*) Rational (*iii*) Real but irrational.
(*iv*) Coincident (*v*) Imaginary (*vi*) Real but irrational.
3. ± 30. **4.** 4. **5.** 0, 3.
21. (*i*) $a=c$ (*ii*) b negative and c positive
(*iii*) b and c positive (*iv*) c negative. (*v*) $b=0$.
22. (*i*) $c=0$. (*ii*) $b=c=0$ (*iii*) $a=0$ (*iv*) $a=b=0$.
23. ∞, ∞.

Exercise VI.

1. $x=3, -\frac{19}{7}$
$y=2, -\frac{13}{7}$.
2. $x=2, -9$
$y=9, -2$.
3. $x=5, \frac{65}{7}$
$y=4, \frac{8}{7}$
4. $x=\frac{7}{3}$
$y=\frac{5}{2}$.
5. $x=$, $\frac{23}{7}$
$y=2, \frac{2}{7}$.
6. $x=-7, 4$
$y=5, -\frac{1}{2}$.
7. $x=-1, \frac{77}{17}$
$y=-7, \frac{21}{17}$.
8. $x=2, \frac{33}{14}$
$y=1, \frac{44}{49}$.
9. $x=3, 5$
$y=5, 3$.
10. $x=7, -2$
$y=2, -7$.
11. $x=\frac{2}{l}, \frac{1}{l}$
$y=\frac{1}{m}, \frac{2}{m}$.
12. $x=\pm 5, \pm 4i$
$y=\mp 4, \pm 5i$.
13. $x=\pm\sqrt{6}$
$y=\pm 4\sqrt{6}$.
14. $x=\frac{-1\pm\sqrt{1585}}{4}$
$y=\frac{1\pm\sqrt{1585}}{6}$.
15. $x=3, 3, -3, -3$
$y=2, -2, 2, -2$.
16. $x=a, a, -a, -a$
$y=b, -b, -b, b$.
17. $x=4, -4$
$y=4, \quad 4$.
18. $x=2$
$y=3$
19. $x=\mp\frac{1}{2}$
$y=\pm 4$.

20. $x=3, 3, 3, 3, -3, -3, -3, -3$
$y=2, -2, -2, 2, 2, -2, -2, 2$
$z=5, 5, -5, -5, -5, -5, 5, 5.$

21. $x=1, 1, 1, 1, -1, -1, -1, -1$
$y=2, -2, -2, 2, 2, -2, -2, 2$
$z=3, 3, -3, -3, -3, -3, 3, 3.$

22. $x=3, \frac{-9}{4}, \frac{3}{4}, \frac{9}{19}$
$y=4, -3, -1. -\frac{12}{19}$
$z=1, \frac{3}{4}, -\frac{1}{4}, \frac{8}{19}.$

23. $x=\pm 3$
$y=\pm 5$
$z=\pm 4.$

24. $x=3, -3, -3, 3$
$y=2, 2, -2, -2.$

25. $x=8, 2$
$y=2, 8.$

26. $x=3\ 2$
$y=\frac{1}{3}, \frac{1}{3}.$

27. $x=1, \frac{1}{2}$
$y=\frac{1}{2}, 1.$

28. $x=5, \frac{3}{4}$
$y=3, -\frac{5}{4}.$

29. $x=12, 4$
$y=4, 12.$

30. $x=2, 1$
$y=1, 2.$

31. $x=9, 4$
$y=4, 9.$

32. $x=36, 1$
$y=1, 36$

33. $x=4, 2, 3+\sqrt{21}. 3-\sqrt{21}$
$y=2, 4, 3-\sqrt{21}, 3+\sqrt{21}.$

34. $x=8, 2$
$y=2, 8.$

35. $x=\pm 6$
$y=\pm 2.$

Exercise VII.

1. $x=8, 4$
$y=4, 8.$

2. $x=6, 5$
$y=5, 6.$

3 $x=1, \frac{1}{9}$
$y=\frac{1}{2}, 1.$

4. $x=10, -10, 9, -9$
$y=-9, 9, -10, 10.$

5. $x=3, 2, -3+\sqrt{3}, -3-\sqrt{3}$
$y=2, 3, -3-\sqrt{3}, -3+\sqrt{3}.$

6. $x=5, 1, 1, -4$
$y=-1, -5, 4, -1.$

7. $x=5, 4, 8+\sqrt{-\frac{9}{4}}, 8-\sqrt{-\frac{9}{4}}$
$y=4, 5, 8-\sqrt{-\frac{9}{4}}, 8+\sqrt{-\frac{9}{4}}.$

8. $x=3, \frac{1}{2}(7\pm\sqrt{-3})$
$y=4, \frac{1}{2}(7\pm\sqrt{-3}).$

9. $x=8, 2$
$y=2, 8.$

10. $x=3, -\frac{3}{2}$
$y=4, -\frac{19}{2}.$

11. $x=4, 5$
$y=5, 4$

12. $x=b, \frac{a^2}{b}$
$y=a, \frac{b^2}{a}.$

13. $x=\frac{a}{2}$
$y=\frac{b}{2}.$

14. $x=a,\ b$
$y=b,\ a.$

15. $x=(\sqrt{a}\pm\sqrt{b})^2$
$y=(\sqrt{a}\mp\sqrt{b})^2.$

Exercise VIII.

1. $x=\pm 6,\ \pm\sqrt{13}$
$y=\pm 7,\ \mp\sqrt{1}$.

2. $x=\pm 4,\ \pm 3\sqrt{3}$
$y=\pm 5,\ \pm\sqrt{3}.$

3. $x=\pm 4,\ \pm 23i$
$y=\pm 3,\ \mp 10i.$

4. $x=\pm 3,\ \pm 3\sqrt{-5}$
$y=\pm 4,\ \pm 2\sqrt{-5}.$

5. $x=3,\ \pm(\sqrt{22}+\sqrt{-5})$
$y=1,\ \pm(\sqrt{22}-\sqrt{-5}).$

6. $x=\pm a,\ \pm 2b\sqrt{\frac{2}{5}}$
$y=\pm 2b,\ \pm a\sqrt{\frac{5}{2}}.$

7. $x=\dfrac{ab^2\pm ab\sqrt{2a^2-b^2}}{b^2-a^2}$
$y=\dfrac{ab^2\mp ab\sqrt{2a^2-b^2}}{b^2-a^2}$.

8. $x=\pm 6,\ \pm 6$
$y=\pm 4,\ \mp 4$

9. $x=\pm 1,\ \pm 2$
$y=\pm 3,\ \pm 1.$

10. $x=\pm 3.\ \pm 8\sqrt{\frac{37}{127}}$
$y=\pm 4,\ \pm 6\sqrt{\frac{37}{127}}.$

Exercise IX.

1. $x=5,\ 5,\ -7,\ -7$
$y=0,\ 3,\ \ 3,\ \ 0.$

2. $x=1,\ -\frac{42}{1}\frac{1}{7}$
$y=-11,\ \frac{1}{1}\frac{1}{87}.$

3. $x=2a,\ -a,\ \frac{1}{2}(5a^2\pm\sqrt{2a-25a^4})$
$y=a,\ -2a,\ \frac{1}{2}(-5a^2\pm\sqrt{2a-25a^4}).$

4. $x=\frac{1}{4}(5\pm\sqrt{13})$
$y=\frac{1}{8}(5\mp\sqrt{13}).$

5. $x=9,\ 4$
$y=4,\ 9.$

6. $x=3,\ -\frac{24}{7}$
$y=4,\ -\frac{32}{7}.$

7. $x=3,\ -2$
$y=2,\ -3.$

8. $x=\frac{1}{2}(3\pm\sqrt{5})$
$y=\frac{1}{2}(1\pm\sqrt{5}).$

9. $x=\pm 1,\ \pm 3,\ \pm 4$
$y=\pm 12,\ \pm 4,\ \pm 3.$

10. $x=3,\ -1,\ 7\pm2\sqrt{13},\ 7\pm\sqrt{10},\ 7\pm\sqrt{-38}$

$y=-1,\ 3,\ 7\mp2\sqrt{13},\ 7\mp\sqrt{10},\ 7\mp\sqrt{-38}.$

11. $x=5,\ 13$
$y=4,\ 12.$

12. $x=1,\ 2,\ 3,\ 5$
$y=5,\ 3,\ 2.\ 1.$

13. $x=4,\ 9$
$y=9,\ 4.$

14. $x=4,\ 2$
$y=2,\ 4.$

15. $x=4,\ -2,\ 1\pm\sqrt{13}$
$y=2,\ -4,\ -1\pm\sqrt{13}.$

16. $x=5,\ 5$
$y=4,\ -4.$

17. $x=3,\ 3,\ -\frac{1}{3},\ -\frac{1}{3}$
$y=3,\ -\frac{1}{3},\ 3,\ -\frac{1}{3}.$

18. $x=0,\ 1,\ \frac{1}{2}\frac{5}{2}$
$y=0,\ 2,\ \frac{9}{2}\frac{1}{2}.$

19. $x=2,\ -\frac{5}{8}\frac{5}{2}\frac{1}{7}$
$y=3,\ \frac{1}{8}\frac{2}{2}\frac{4}{7}.$

20. $x=\pm\frac{\sqrt{3}}{3}(2b-a),\ \pm ai$
$=\pm\frac{\sqrt{3}}{3}(2a-b),\pm bi$

21. $x=\pm3$
$y=\pm2.$

22. $x=\pm8$
$y=\pm5$
$z=\pm3.$

23. $x=\frac{1}{7}^{2}$
$y=\frac{1}{5}^{2}$
$z=-12.$

24. $x=\pm3$
$y=\pm5$
$z=\pm7.$

25. $x=\pm2$
$y=\ \ 0$
$z=\pm1.$

Miscellaneous Questions I

1. (*i*) $3^n,\ 5^n.$ (*ii*) $-4.$ (*iii*) $2,\ \frac{1}{4},\ \frac{9\pm i\sqrt{31}}{8}$
(*iv*) $3,\ \frac{1}{3}^{7}.$ **2.** 1 and $-4.$ **4.** 6.

5. $\pm3\sqrt{2},\pm\sqrt{-10}.$ **13.** $27,-125.$
14. (*i*) $1-l-m+l^2-lm+m^2.$ (*ii*) $l^3-3m.$
15. $w,w^2,\pm\sqrt{5}.$ **16.** $x=\pm1,\pm i\frac{3}{2}$; $y=\pm3,\mp i2.$
19. $3ax^2-3a^2x+a^3-b^3=0.$
21. (*i*) $x=7,-1,3\pm\sqrt{-3}$ (*ii*) $x=5,\ \frac{1}{8}^{4}\frac{5}{}$
$y=1,-7,-3\pm\sqrt{-3}.$ $y=4,\ \frac{1}{8}^{4}\frac{3}{}.$
(*iii*) $x=2,3,6,14$ (*iv*) $x=3,\ 3w,\ 3w^2$
$y=3,2,1,$ $y=2,\ 2w,\ 2w^2$ $w=20,\ 20w,\ 20w^2$
$z=11,\ 11w,\ 11w^2$

23. $7\pm3\sqrt{5}.$

Exercise X.

1. $\frac{1}{x+1}+\frac{1}{x+2}.$ **2.** $\frac{1}{x+1}+\frac{2}{x-1}.$

3. $\dfrac{4}{1-3x} - \dfrac{5}{1-2x}$.

4. $\dfrac{5}{3x+2} - \dfrac{4}{4x-5}$.

5. $\dfrac{1}{3}\left[\dfrac{1}{3-x} - \dfrac{1}{3+2x}\right]$.

6. $\dfrac{1}{x+1} + \dfrac{1}{x-1} - \dfrac{1}{x}$.

7. $3x+1+\dfrac{3}{x+2} - \dfrac{1}{2x-3}$.

8. $\dfrac{1}{6(x+1)} - \dfrac{1}{2(x-1)} + \dfrac{4}{3(x-2)}$.

9. $2+\dfrac{7}{3x-5} - \dfrac{5}{4x+3}$.

10. $\dfrac{1}{a+b}\left[\dfrac{pa+q}{x-a} + \dfrac{pb-q}{x+b}\right]$.

11. $\dfrac{2}{x-1} + \dfrac{3}{x-2} - \dfrac{4}{x-3}$.

12. $\dfrac{1}{2}\left[\dfrac{1}{x-4} + x\dfrac{1}{-3} - \dfrac{1}{x-2} - \dfrac{1}{x-1}\right]$.

13. $\dfrac{a^2}{(a-b)(a-c)} \cdot \dfrac{1}{1-ax} + \dfrac{b^2}{(b-c)(b-a)} \cdot \dfrac{1}{1-bx} + \dfrac{c^2}{(c-a)(c-b)} \cdot \dfrac{1}{1-cx}$.

14. $\dfrac{6}{x^2+3} - \dfrac{4}{x^2+2}$

15. $1-\dfrac{1}{x^3+2} - \dfrac{2}{x^3+3} - \dfrac{3}{x^3+4}$.

Exercise XI.

1. $\dfrac{1}{x-1} - \dfrac{1}{x+2} - \dfrac{3}{(x+2)^2}$.

2. $x-2-\dfrac{17}{16(x-3)} + \dfrac{17}{16(x+1)} - \dfrac{11}{4(x+1)^2}$.

3. $\dfrac{1}{x-1} - \dfrac{1}{x+1} + \dfrac{3}{(x+1)^2} - \dfrac{3}{(x+1)^3} + \dfrac{2}{(x+1)^4}$.

4. $\frac{3}{x-1}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$.

5. $\frac{-6}{2+3x}+\frac{2}{1+x}+\frac{3}{(1+x)^2}$.

6. $\frac{4}{25(1+x)}+\frac{89}{100(1-4x)}+\frac{39}{20(1-4x)^2}$.

7. $\frac{2}{(1-x)^3}-\frac{3}{(1-x)^2}+\frac{2}{1-x}$.

8. $\frac{5}{x+1}+\frac{4}{(x+1)^2}+\frac{3}{x-1}-\frac{4}{(x-1)^2}-\frac{5}{(x-1)^3}$.

9. $\frac{1}{x-2}-\frac{1}{(x-2)^2}+\frac{1}{(x-2)^3}-\frac{1}{x-3}+\frac{1}{(x-3)^2}$.

10. $\frac{-5}{3(x^2-2)}-\frac{4}{x^2-2}-\frac{1}{3(1-2x^2)}$.

Exercise XII.

1. $\frac{1}{3(x-1)}+\frac{2x+1}{3(x^2+x+1)}$.

2. $\frac{3}{x^2+1}-\frac{2}{x+3}$.

3. $\frac{1}{2(1-x)^2}+\frac{1}{2(1+x^2)}$.

4. $\frac{5-2x}{2(1+x+x^2)}+\frac{3}{2(1+x)}$

5. $\frac{3}{2(x-1)}-\frac{3x-1}{2(x^2+1)}$.

6. $\frac{3(3-2x)}{(3-3x+x^2)^2}-\frac{2}{3-3x+x^2}$.

7. $\frac{2}{x^2-x+1}-\frac{1}{x^2+x+1}$.

8. $\frac{1}{1+x}-\frac{2}{1+x^2}+\frac{3}{(1+x^2)^2}$.

9. $\frac{1}{x^2+x+1}-\frac{1}{x^2+x+2}$.

10. $\frac{2x+1}{x^2-x+1}-\frac{2}{x-1}-\frac{1}{(x-1)^2}$.

11. $x-\frac{16x}{3(x^2+4)}+\frac{x}{3(x^2+1)}$.

12. $\frac{1}{x}-\frac{x}{1+x^2}-\frac{x}{(1+x^2)^2}-\frac{x}{(1+x^2)^3}$.

Exercise XIII.

1. 19, 49, $2n+3$. **2.** -73, -145, $31-4r$.
3. 9th, 13th, nth, $(2r-2)$th. **4.** 27 69, $6p-3$.
5. $2+5+8+11+$ ..... **6.** $-8\frac{1}{2}\frac{7}{8}$.
7. 4th. **8.** 11, 76, $5m-9$.
9. (*i*) 306; (*ii*) 2265; (*iii*) $202\frac{1}{2}$;
(*iv*) 225; (*v*) 0; (*vi*) -2138.5;
(*vii*) $\frac{210}{x}$; (*viii*) $\frac{33n-n^2}{24}$; (*ix*) $\frac{n(n-1)}{2}$;
(*x*) $\frac{n-1}{2}$. **10.** (*i*) -500; (*ii*) $-\frac{1}{2}(18+135\sqrt{3})$
(*iii*) $\frac{n^2-n+2}{2}$; (*iv*) 2325; (*v*) $432\frac{1}{7}$.
11. (*i*) 1830; (*ii*) 2025; (*iii*) $n(n+1)$.
12. 20+. **13.** $\frac{a+c}{2}$. $\frac{2c+b-3a}{b-a}$. **14** (*i*) 16;
(*ii*) 23; (*iii*) 9 or 19. **15.** 43.
16. 19. **17.** 810. **18.** $n(2n-29)$.
19. 9,4. **20.** $a+b$, $2a$. **21.** $n(2n-1)$
22. (*i*) $\frac{n}{2(1-x)}\{2+(n-3)\sqrt{x}\}$; (*ii*) $15(a^2-12ax+x^2)$;
(*iii*) $\frac{n}{a+b}\left[na-\frac{n+1}{2}b\right]$; (*iv*) $\frac{3n-1}{2}$;
(*v*) $\frac{1}{2}(x^2-2xy-3y^2+3x+y)$. **23.** 9.
24. Rs. 110, Rs. 2070. **25.** 16.

Exercise XIV.

1. 7, 10, 13, 16, 19, 22. **2.** 19,17,15,....., -3.
3. m^2-m+1, m^2-2m+2, m^2-3m+3, ..., m.
4. $(5a-7b)+2\frac{a+b}{r+1}$, $(5a-7b)+4\frac{a+b}{r+1}$, $(5a-7b)+6\frac{a+b}{r+1}$,....., $(5a-7b)+2r\frac{a+b}{r+1}$.
6. 26. **9.** 0.

Exercise XV

1. 1, 3, 5. **2.** 7, 11, 15. **3.** 7, 9, 11, 13.
4. 1, 2, 3, 4. **5.** 4, 6, 8, 10. **11.** $9\frac{2}{5}$. **12.** 429.
13. 9 : 23. **14.** $-11:8$. **15.** 23 : 17 ; 37 : 23.
17. $\frac{1}{2}n(n^2+1)$. **18.** $n^3+(n-1)^3$.
20. 7, 6. **34.** -20. **35.** $\frac{3}{2}n(n-1)$.

Exercise XVI

1. 2. **2.** $\frac{7}{85}\frac{}{81}$. **3.** $\frac{4}{9}$, $\frac{128}{2187}$.

4. $-\frac{1}{256}$, $\frac{2}{(-2)^{n-1}}$. **5.** 1, 16. **6.** 9th.

7. 6th. **8.** 729, 1093. **9.** $\frac{1054699}{17717}$, $-\frac{20488}{17717}$.

10. $\frac{14762}{65661}$. **11.** (*i*) 63 ; (*ii*) $\frac{58025}{21887}$; (*iii*) $\frac{55767}{31250}$; (*iv*) $31+15\sqrt{2}$; (*v*) $\frac{1062}{3}82$ $(1-\sqrt{3})$ (*vi*) $\frac{1+\sqrt{2}}{\sqrt{2}}-\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^{n-1}$; (*vii*) $\frac{1}{2}(2+\sqrt{2})\{1-(\sqrt{2}-1)^n\}$;

(*viii*) $\frac{a+b}{(a-b)^{n-2}}\left\{\frac{(a-b)^n-1}{a-b-1}\right\}$; (*ix*) $l^p\left\{\frac{l^{nq}-1}{l^q-1}\right\}$;

(*x*) $p^2\,\frac{p^{2r}-1}{p^2-1}+pq\,\frac{p^rq^r-1}{pq-1}$; (*xi*) $\frac{n(n^p-1)}{n-1}+\frac{mp}{2}(p+1)$; (*xii*). $xn^2+\frac{1}{6}\{1-(-\frac{1}{2})^n\}$.

12. (*i*) $\frac{1}{2}$; (*ii*) $\frac{25}{12}$; (*iii*) $\frac{4}{15}$; (*iv*) $\frac{2}{3}$;

(*v*) $\frac{\sqrt{3}}{2}$; (*vi*) 1728 ; (*vii*) $6\frac{3}{4}$; (*viii*) $4\frac{1}{3}$;

(*ix*) $\frac{1}{2}(5+3\sqrt{3})$; (*x*) $\frac{1}{1+3a}$; (*xi*) $\frac{17}{12}$;

(*xii*) $\frac{ax+b}{x^4-1}$. **13.** (*i*) $\frac{1}{32}(126+127\sqrt{2})$;

(*ii*) $\frac{15}{31}\frac{62}{25}$; (*iii*) $\frac{27}{20}$; (*iv*) 5.

14. (*i*) $\frac{7}{19}\frac{}{85}$; (*ii*) $\frac{655}{900}$; (*iii*) $\frac{11}{4}\frac{161}{965}$. **15.** $\frac{8}{3}$. **16.** 6.

17. $2+6+18+54+\ldots\ldots$ **18.** $\frac{1}{3}$.

19. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$

Exercise XVII

1. $\pm 27, 18, \pm 12, 8, \pm 5\frac{1}{3}$. **2.** $\frac{27}{8}, \frac{9}{4}, \frac{3}{2}, 1$.

3. $+\frac{2}{3}, \sqrt{\frac{2}{8}}, \pm 1, \sqrt{\frac{2}{3}}, \pm\frac{3}{2}$. **4.** $-9, \frac{27}{4}, -\frac{81}{16}, \frac{243}{64}$, $\frac{729}{256}, \frac{2187}{1024}$. **7.** 16, 4. **8.** $-\frac{1}{2}$.

Exercise XVIII

1. (*i*) $\frac{3}{2}n(n+1)+\frac{3}{2}(3^n-1)$; (*ii*) $n(x+2)+2(2^n-1)$;
(*iii*) $4-\frac{1}{2^{n-4}}$. **2.** (*i*) $\frac{5}{8}\frac{9}{1}(10^n-1)-\frac{5n}{9}$;
(*ii*) $\frac{4}{8}\frac{9}{1}(10^n-1)-\frac{4n}{9}$; (*iii*) $\frac{7n}{9}-\frac{7n}{81}(1-\frac{1}{10^n})$;
(*iv*) $2(n-1)+\frac{1}{2^{n-1}}$. **3.** 5, 15, 45.
4. 3, 15, 75. **5.** 4, 8, 16. **6.** 2, 5, 8. **7.** 2, 3, 4.
8. $2\frac{1}{2}$, 5, 10. **9.** 1, 3, 5, 7. **10.** 1, 2, 4, 8.
13. $\frac{b^2-ac}{a+c-2b}$. **16.** $\frac{3}{5}$. **23.** $y-\frac{(y-x)^n}{y^{n-1}}$.
25. $3x$ ft.

Exercise XIX

1. 4. **2.** $\frac{3}{1}\frac{}{6}$. **3.** $2\frac{1}{9}$. **4.** $\frac{1}{1}\frac{6}{9}$, $\frac{8}{1}\frac{}{7}$, $\frac{1}{4}\frac{6}{9}$.
5. $\frac{1}{9}$, $\frac{1}{1}\frac{1}{2}$, $\frac{1}{1}\frac{1}{5}$, $\frac{1}{1}\frac{}{8}$, $\frac{1}{2}\frac{1}{1}$. **6.** $\frac{1}{2}$. **7.** $-\frac{3}{3}\frac{}{1}$.
8. $\frac{4}{7-3n}$. **9.** $\frac{3}{1}\frac{}{6}$.
10. $\frac{4}{3}\frac{}{1}+\frac{1}{7}+\frac{4}{2}\frac{}{5}+\frac{2}{1}\frac{}{1}+\frac{1}{1}\frac{1}{9}+\frac{1}{4}+\frac{4}{1}\frac{}{3}+\frac{2}{5}+\frac{4}{7}$. **11.** 6, $\frac{3}{2}$.
12. -1. **13.** $\frac{4}{2}\frac{}{5}$ and $\frac{2}{1}\frac{}{5}$. **14.** 8, 72.
15. 12, 3. **16.** $\frac{mn}{m+n}$.

Exercise XX

1. $1-\frac{n+1}{3^n}$. **2.** $\frac{3}{1}\frac{5}{6}$. **3.** $\frac{1}{4}\frac{3}{9}$. **4.** $4-\frac{n+2}{2^{n-1}}$.
5. $\frac{5}{1}\frac{}{8}$. **6.** $(n-1)2^n+1$. **7.** $\frac{35}{16}-\frac{12n+7}{16.5^{n-1}}$; $\frac{35}{16}$.
8. $\frac{1}{1-x}+\frac{3x(1-x^{n-1})}{(1-x)^2}-\frac{(3n-2)x^n}{1-x}$.
9. $\frac{1}{1-x^2}+\frac{2x^2(1-x^{2n-2})}{(1-x^2)^2}-\frac{(2n-1)x^{2n}}{1-x^2}$; $\frac{1+x^2}{(1-x^4)^2}$.
10. $\frac{1+x+x^2}{(1+x)^2}$. **11.** $\frac{1+6x+x^2}{(1-x)^3}$. **12.** $1\frac{1}{2}$.
13. $\frac{n}{1+x}+\frac{x(1-x^n)}{(1-x)^2}$. **14.** $\frac{1}{(1-r)(1-ar)}$.

15. $\dfrac{a^2}{(a-x)(a-x^2)}$.

Exercise XXI

1. $\frac{2}{3}n(n+1)(2n+1)$. **2.** $\frac{1}{3}n(4n^2-1)$. **3.** $p^2(2p^2-1)$.

4. (*i*) $\frac{1}{3}n(n+1)(n+2)$; (*ii*) $n(3n^2+6n+1)$;

(*iii*) $\frac{n}{3}(4n^2+18n+23)$; (*iv*) $\frac{n}{3}(n^2+12n+29)$;

(*v*) $n(2n^3+8n^2+7n-2)$; (*vi*) $\frac{n}{3}(16n^2+12n-1)$;

(*vii*) $\frac{n}{12}(n+1)(n+2)(3n+5)$;

(*viii*) $\frac{n}{6}(n+1)(3n^2+19n+32)$.

5. (*i*) $\frac{n}{6}(2n^2+9n+13)$. (*ii*) $\frac{n}{3}(n^2-7)$.

(*iii*) $\frac{n}{3}(n+2)(2n-1)$; (*iv*) $\frac{3}{4}n^2(n+1)^2+2n$;

(*v*) $2n+1+n(n+1)(2n+1)-2$; (*vi*) $\frac{n}{2}(n+1)^2(n+2)$.

7. $\frac{n}{6}(n+1)(n+2)$. **8.** $\frac{n}{12}(n+1)^2(n+2)$.

9. $na^2+n(n-1)ad+\frac{n}{6}(n-1)(2n-1)d^2$.

Miscellaneous Questions II

1. $\frac{n}{200}(n+139)$. **2.** 16. **3.** 3, 5, 7, 9.

4. $(n^2-2\pm n\sqrt{n^2-4}):2$ **5.** $9\frac{1}{4}\frac{5}{3}\frac{1}{2}$. **6.** $\dfrac{60}{16-n}$.

7. 4 or 10. **8.** 8, 5. **9.** $\frac{1}{12}(n+1)(2n^2+n+3)$.

11. $\frac{7}{3}\frac{5}{2}$. **12.** 7. **13.** (*i*) $-\frac{n}{6}(2n^2+3n+5)$;

(*ii*) $\dfrac{1}{1+x}+\left(\dfrac{x}{1+x}\right)^2$; (*iii*) $-n(2n+1)$.

14. $\frac{n}{6}(n+1)(n+2)$. **16.** $\frac{n}{2}(n-1)(2n-1)$. **17.** 7.

18. 1, 3, 5, 7, 9. **19.** (*i*) $\frac{n}{12}(n+1)(3n^2+19n+37)$,

(*ii*) $\frac{n}{2}(n+1)(n+2)(3n+1)$. **22.** $(-1)^{n-1}(+n^3-3n)$.

28. $\frac{a^n-1}{a-1} \cdot \frac{b^n-1}{b-1} \cdot \frac{1}{b^{n-1}}$. **31.** $\frac{1}{1+n} - \frac{1}{1+2n}$.

32. $\frac{1}{n(1-n)}\left\{\frac{1}{1+n^{n+1}} - \frac{1}{1+n}\right\}$. **33.** $\frac{1}{1+n}$.

34. $\frac{x}{1-x}$.

Exercise XXII

1. (*i*) 60, 90, 20160, 18. (*ii*) 5040, 1320, ৯$\frac{1}{4}$ক, ক$\frac{1}{5}$ক. 6435, $\frac{7}{15}$, 4900. **3.** (*i*) 6. (*ii*) 8. (*iii*) 31. (*iv*) 9, (*v*) 5. (*vi*) 8. **5.** 20. **6.** 60. **7.** $\lfloor\underline{10}$.

8 $\lfloor\underline{6}$. **9.** $\lfloor\underline{6}$. **10.** $\lfloor\underline{6}$, 3 $\lfloor\underline{5}$.

11. 360, 12. **12.** 840. **13.** 1099. **14.** 13440.

15. 151200. **16.** $\lfloor\underline{5}$. **17.** (*i*) 720 (*ii*) 4320.

18. 1440. **19** $(r-2)\ \lfloor\underline{r-1}$ **20.** $\lfloor\underline{8}$,

(*i*) $\lfloor\underline{7}$. (*ii*) $\lfloor\underline{6}$. (*iii*) 2 $\lfloor\underline{7}$. (*iv*) 6 $\lfloor\underline{6}$.

(*v*) 50 $\lfloor\underline{6}$. (*vi*) 12 $\lfloor\underline{6}$. **21.** $\lfloor\underline{5}$. **22.** $\frac{1}{2}\lfloor\underline{8}$.

23. $\lfloor\underline{5}\ \lfloor\underline{6}$. **24.** 96, 24. **25.** 13699.

26. 7^4. **27.** $\lfloor\underline{5}\ (\lfloor\underline{4})^3(\lfloor\underline{3})^2$.

28. (*i*) 720, (*ii*) 240.

Exercise XXIII

1. 20, 3003, 816, 48, $\left(\frac{\lfloor\underline{2r}}{\lfloor\underline{r}}\right)^2$. **2.** (*i*) 13; (*ii*) 7; (*iii*) 6; (*iv*) 39; (*v*) 8; (*vi*) 8.

3. 54264, 20349. **4.** $n=34$. $r=14$.

7. 30. **8.** $\frac{1}{6}n(n-1)(n-2)$. **9.** $\frac{1}{2}n(n-3)$.

10. (*i*) 105 (*ii*) 455. (*iii*) 90. **11.** 756.

12. 700. **13.** (*i*) 56. (*ii*) 70. **14.** (*i*) 2002.

(*ii*) 364. **15.** (*i*) 144. (*ii*) 806.

16. $\frac{\lfloor\underline{n-r}}{\lfloor\underline{n-2}\ \lfloor\underline{r-2}}$. **17.** (*i*) 84. (*ii*) 126.

18. 378,000. **19.** 11520. **20.** 56, 21.

21. 84. **23.** $\frac{\lfloor 90}{\lfloor 50 \, \lfloor 40}$ **24.** $\frac{\lfloor 21}{(\lfloor 7)^3}$.

25. 27720.

Exercise XXIV

1. 180. **2.** 43,243,200. **3.** n^2. **4.** 7^5.
5. 127. **6.** 1727. **7.** 779. **8.** 103680.
9. 6300. **10.** 14400. **11.** 30240.
12. 7560. **13.** 105. **14.** 5936.
15. 728. **18.** 4^6. **19.** 1490. **21.** 35.
22. $^{44}C_5$. **23.** 26, 136. **24.** 255. **25.** 5039.
26. $\frac{\lfloor 990}{50 \, \lfloor 940}$. **27.** 18. **28.** $(2^{12}-1)(2^6-1)(2^5-1)$.

30. 35. **31.** 2002. **33.** $\frac{\lfloor 2n-p-q}{n-p \, \lfloor n-q} (\lfloor n)^2$. **35.** 432.

36. 78. **37.** 15. **38** 11280. **39.** $(\lfloor 4)^{13}$.

Exercise XXV

1. (*i*) $x^5+10x^4a+40x^3a^2+80x^2a^2+80xa^4+32a^5$.

(*ii*) $1-7x+21x^2-35x^3+35x^4-21x^5+6x^6-x^7$.

(*iii*) $x^6+6x^4+15x^2+20+\frac{15}{x^2}+\frac{6}{x^4}+\frac{1}{x^6}$.

(*iv*) $\frac{32}{243}x^5+\frac{40}{27}x^3+\frac{20}{3}x+\frac{15}{x}+\frac{135}{8x^3}+\frac{243}{32x^5}$.

(*v*) $1-18x^2+135x^4-540x^6+1215x^8-1458x^{10}+729x^{12}$

(*vi*) $x^{14}+14x^{11}y+84x^8y^2+280x^5y^3+560x^2y^4+672\frac{y^5}{x}+448\frac{y^6}{x^4}+128\frac{y^7}{x^7}$.

(*vii*) $a^2-4a^{\frac{3}{2}}b^{\frac{1}{2}}+6ab-4a^{\frac{1}{2}}b^{\frac{3}{2}}+b^2$.

(*viii*) $2187a^7+1701a^5+567a^3+105a+\frac{35}{3a}+\frac{7}{9a^3}+\frac{7}{243a^5}+\frac{1}{2187a^7}$.

(*ix*) $\frac{x^{16}}{256}-\frac{x^{12}}{8}+\frac{7x^8}{4}-14x^4+70-\frac{224}{x^4}+\frac{448}{x^8}-\frac{512}{x^{12}}+\frac{256}{x^{16}}$.

2. (*i*) 82. (*ii*) $378x+63x^3+126x^5+2x^7$.

(*iii*) $32x^5-40x^3a^2+10xa^4$. **3.** $1088640\frac{1}{x^2}$.

4. $\frac{55}{9}x^8$. **5.** $\frac{1771}{10}\cdot\frac{1}{p^{13}}$. **6.** $\frac{1820}{81}x^8$.

7. $-\frac{\lfloor 14}{(\lfloor 7)^2}x^7y^7$. **8.** $\frac{567}{2\sqrt{2}}\frac{x^5}{a^4}$; $\frac{567\sqrt{3}}{2}\frac{x^4}{a^5}$.

9. $\frac{13.12^{12}}{3}$. **10.** $210a^{16}b^6$. **11.** 8; no term containing x^9. **12.** $\frac{7}{178}$ **16.** 7. **19** $(1+2)^7$.
20. 6.

Exercise XXVI.

1. (*i*) 3rd; (*ii*) 8th; (*iii*) 22nd; (*iv*) 5th and 6th.
2. (*i*) $_{14}C_7$; (*ii*) 4860; (*iii*) 56229888; **3.** $r+1$.
4. $(-1)^{r-1}\frac{\lfloor 2n}{\lfloor r-1\,\lfloor 2n-r+1}x^{2n-2r+2}$;

$(-1)^{r-1}\frac{2n}{\lfloor r-1\,\lfloor 2n-r+1}x^{2r-2n-2}$; $(-1)\frac{2n}{(\lfloor n)^2}$.

5. (*i*) $2^{12}-1$; (*ii*) $2^{19}-1$; (*iii*) $2(2^{21}-1)$.
6. (*i*) $n2^{n-1}$; (*ii*) $(n+2)2^{n-1}$; (*iii*) $\frac{2n}{n-1\,\lfloor n+1}$.

(*iv*) $\frac{2n}{\lfloor n-2\,\lfloor n+2}$; (*v*) $\frac{\lfloor 2n}{\lfloor n-3\,\lfloor n+3}$;

(*vi*) $\frac{\lfloor 2n}{\lfloor n-r\,\lfloor n+r}$; (*vii*) $\frac{n(n+1)}{2}$.

7. $1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$.
8. $1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$.
9. $2\,{}^nC_2+6\,{}^nC_3+4\,{}^nC_4+{}^nC_5$.

Exercise XXVII.

1. $1+6x+24x^2+80x^3+240x^4+\ldots\ldots$
2. $1-\frac{1}{5}x-\frac{2}{25}x^2-\frac{6}{125}x^3-\ldots$

3. $1+x+\frac{1}{6}x^2-\frac{1}{54}x^3+\ldots\ldots\ldots$

4. $1-\frac{7}{3}x^2+\frac{14}{9}x^4-\frac{14}{81}x^6-\frac{7}{243}x^8-\ldots\ldots\ldots$

5. $1+\frac{x}{a}+\frac{3}{2}\cdot\frac{x^2}{a^2}+\frac{5}{2}\cdot\frac{x^3}{a^3}+\ldots\ldots\ldots$

6. $(3a)^{\frac{2}{3}}\left[1-\frac{4}{9}\frac{x}{a}-\frac{4}{81}\frac{x^2}{a^2}-\frac{32}{2187}\frac{x^3}{a^3}-\ldots\right]; x<\frac{3}{2a}$.

7. $\frac{a^4}{81}+\frac{8a^5x}{243}+\frac{40a^6x^2}{729}+\frac{160a^7x^3}{2187}+\frac{560a^8x^4}{6561}+\ldots\ldots; x<\frac{3a}{2}$

8. $3\left[1+\frac{1}{3}\frac{x^2}{a^3}+\frac{2}{9}\frac{x^4}{a^6}+\frac{14}{81}\frac{x^6}{a^9}+\frac{35}{243}\frac{x^8}{a^{12}}+\ldots\ldots\right]$;

$$\frac{1.4.7.\ldots\ldots(3r-2)}{3^{r-1}\,|\,\underline{r}}\left[\frac{x^2}{a^3}\right]^r.$$

9. $-945a^{-24}b^3$; $-30618a^{-30}b^5$.

10. $\frac{1.5.9\ldots(4r-3)}{4^r\,|\,\underline{r}}x^r$. **11.** $\frac{7.9.11\ldots(2r+5)}{|\,\underline{r}}x^r$.

12. $-1848x^{13}$.

13. $\frac{15\times17}{2^3}a^{-\frac{99}{5}}x^{-10}$; $\frac{15.17.19.21.23}{2^5\,|\,\underline{5}}a^{-27}x^{-\frac{5}{2}}$;

$$\frac{15.17.19\ldots(2r+11)}{2^{r-1}\,|\,\underline{r-1}}a^{-\frac{12r+63}{5}}x^{\frac{5r-35}{2}};$$

14. $\frac{(n+1)(2n+1)(3n+1)\ldots(nr-2n+1)}{n^{r-1}\,|\,\underline{r-1}}a^{\frac{n(r-1)+1}{n}}x^{r-1}$;

$x<a^{\frac{24}{55}}$.

15. $\frac{5}{4}$. **16.** $\frac{79}{55}$. **17.** 13th. **18.** $\frac{21.19.17.15}{3^4\,|\,\underline{4}}$.

19. $-\frac{26}{6255}x^4$. **20.** $-\frac{5.2^5}{21^3}x^3$.

21. $-\frac{14.11.8.5.2}{|\,\underline{6}}(7a)^{-\frac{4}{5}}x^6$. **22.** 5th. **23.** 19.24.4.

25. $(-1)^{r+1}$. **26.** $\frac{1}{2}(3n^3+n^2+12n+4)$.

27. $n(n+1)(n+2)\ldots(n+r-1)\,\frac{x^r}{r}$.

28 $(-1)^r 3n(3n+1)(3n+2)\ldots(3n+r-1)\,\frac{x^r}{r}$.

31. 5th ; $+ve$ **32.** 5th ; $+ve$. **33.** 12th ; $-ve$.
34. All terms positive.

Exercise XXVIII

1. $1-\frac{1}{3}x$. **2.** $1-\frac{3}{2}\frac{5}{4}x$. **3.** $(\frac{4}{3}\frac{4}{7})^{\frac{1}{6}}$.
4. x. **5.** $1-\frac{4}{1}\frac{7}{6}\frac{7}{0}x$. **6.** $\frac{1}{2}+\frac{6}{1}\frac{3}{5}\frac{5}{1}\frac{5}{2}x$.
7. 1.99778. **8.** 7.0068. **9.** 5.03968.
10. 1.97438. **11.** 00795. **12.** 1.1705.
15. 32. **16.** $(\frac{4}{7})^{\frac{1}{3}}$. **17.** $(\frac{4}{7})^{\frac{5}{12}}$. **18.** $\sqrt{3}$.
19. $4^{\frac{1}{3}}$. **20.** $\frac{1}{2}(\frac{4}{3}\frac{9}{7})^{\frac{9}{11}}$. **21.** $\frac{5\sqrt{2}}{7}$. **22.** $2^{\frac{1}{3}}$.
23. $2\sqrt{2}$. **26.** $\frac{1}{2}\{(-1)^{r-1}-3\}$.
27. $\frac{1}{3}\{(-2)^{r+1}-1\}$. **28.** $1+(-1)^n$.

Miscellaneous Questions III.

1. $\frac{5}{1}\frac{6}{6}\frac{7}{6}$. **2.** n. **5.** $(1-\frac{1}{9})^{-\frac{7}{2}}$.
7. $\frac{5.9.13.17.21.25.29.33}{8}(\frac{1}{1}\frac{5}{6})^8$. **10.** 3.002.
11. 1. **13.** 10.0332. **14.** 1.059.
15. $\sqrt{\frac{3}{2}}$. **16.** 3003. **17.** $3n+1$.
18. $a=3, b=1$. **21.** 0. **22.** $(\frac{10}{7})^{\frac{1}{3}}$.
23. $19+18r-r^2$.

Exercise XXIX
Miscellaneous

1. (i) $0, \frac{c(a^2+b^2)-(a^3+b^3)}{c(a^2+b^2)+c^2(a+b)}$; (ii) $x=\pm 25,\quad y=\pm 9$.

2. 126. **3.** (i) 13. (ii) $x=0, \frac{a+b+c}{bc}$,

$y=0, \frac{a+b+c}{ca}$, $z=0, \frac{a+b+c}{ab}$.

8. $\frac{1}{9}$. **9.** $(-2)^{n-3}(n-8)$.

10. $(-1)^r \frac{\lfloor n+r-1}{\lfloor n-1 \rfloor r} x^r$; $\frac{\lfloor 2n}{(\lfloor n)^2}$. **12.** 5.01298.

13. 0, $\frac{2^m-1}{2^m+1}a$. **14.** 1, 3, 5, 7. **16.** (*i*) 0, $\frac{2ab}{b-a}$;
(*ii*) $x=3, 1$; $y=1, 3$. **17.** $3r+1$.
19. 127. **20.** (*i*) $-5, 3$; (*ii*) $x=0, \pm 2$. $y=0, \pm 1$
$z=0, \pm 3$. **21.** (*i*) $\pm\sqrt{a+b}$; (*ii*) $x=1, \frac{14}{8}$,
$y=3, \frac{15}{7}$. **22.** 1, 9. **23.** -1; $\frac{1}{8}$; $-\frac{1}{0}$.
24. (*i*) 7; (*ii*) 5. **25.** 2002. **26.** 4320.
29. 300. **33.** 9. **35.** $(\frac{2}{5})^{\frac{1}{3}}$.
36. (*i*) ± 5; (*ii*) $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(a-b)$.
39. 12 men; £14, 18s. **40.** 160. **42.** 3600.
43. 1·0066, **45.** $1\frac{7}{13}$. **50.** (*i*) p^2-q.
(*ii*) $p(3q-p^2)$; (*iii*) $(p^2-q)(p^2-3q)$;

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1925

1. $\pm 4a$; $p^3=q+q^2+3pq$. **2.** (*i*) $\frac{n(n-1)}{2}$.

(*ii*) $\frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\left\{\frac{a+(n-1)d}{1-r}\right\}r^n$.

(*iii*) $2^{n+1}-2-n$. **3.** (*a*) ${}^{n-3}C_r$; (*b*) 9.9833.

1926

1. (*i*) $\frac{a^2+4\pm\sqrt{a^4+4a^2+16}}{2a}$; (*ii*) $5x^2-10x+1=0$; $x=0$.

2. (*i*) 1116. (*ii*) $\frac{1-r^n}{(1-r)^2}-\frac{nr^n}{1-r}$;
(*iii*) $\frac{1}{6}n(n+1)(3x-2)$. **3.** 1.1705.

1927

1. $x^2+b\left(\frac{1}{a}+\frac{1}{c}\right)x+\left(\frac{a}{c}+\frac{c}{a}+\frac{b^2}{ac}-2\right)=0$.

2. $\frac{n^2(n+1)^2}{4}$. **4.** 3.2161.

1928

1. (*i*) $b^2=ac$. (*ii*) $b^2>ac$; $a=b=c$. **2.** $\frac{75}{52}$.

3. 6 $\frac{7}{}$. **4.** $1+\frac{1}{2}x+\frac{3}{8}x^2-\frac{3}{16}x^3+\frac{3}{128}x^4+\frac{15}{256}x^5+\ldots$

when x lies between $\frac{\sqrt{5}-1}{2}$ and -1.

1929

1. $\frac{-p\pm\sqrt{p^2-4q}}{2}$. **3.** (*i*) $\frac{n(2n+1)(4n+1)}{3}$.

(*ii*) $\frac{1}{1+x}+\left(\frac{1}{1+x}\right)^2$. **4.** $(1-\frac{3}{4})^{-\frac{5}{2}}$.

1930

1. (*i*) x infinite. (*ii*) 2, $\frac{1}{2}$. **2.** $\frac{n}{3}(4n^2-1)$.

3. 19. **4.** (*i*) $\frac{1-x^{n+1}}{1-x}$ when $x<1$.

(*ii*) $(1-\frac{1}{9})^{-\frac{7}{2}}$.

1931

1. (*i*) a, b, c positive; (*ii*) 27, -125. **2.** (*i*) $(2+3)^6$.

3. 2, 3; 3, 2. **5.** (*i*) $-n(2n+1)$; (*ii*) $\frac{1+x-x^2}{1-x^2}$.

1932

1. 10, 21. **2.** (*a*) $\frac{a(1-r^{n+1})}{1-r}$; (*b*) 7.

4. (*i*) 996005996001; (*ii*) 2.002; $\frac{1}{1+x}+\left(\frac{x}{1+x}\right)^2$.

1933

2. (*i*) $\left\{\frac{n(n+1)}{2}\right\}^2$; (*ii*) 8, 5. **4.** 1, 2, 4, 8.

1934

1. (*i*) 3, $\frac{1}{3}$; (*ii*) $\frac{13}{14}$, $\frac{1}{2}$. **2.** (*a*) (*i*) $3n^3+6n^2+n$;

(*ii*) $\frac{1}{2}-\frac{1}{3n+2}$ (*b*) $\frac{2}{9}\frac{3}{9}\frac{4}{0}$. **3.** 10.080 when A and σ are similar. **4.** (*b*) 1.987; (*c*) $\frac{f(a)}{(a-b)(a-c)}\cdot\frac{1}{x-a}$

$+\frac{f(b)}{(b-a)(b-c)}\cdot\frac{1}{x-b}+\frac{f(c)}{(c-a)(c-b)}\cdot\frac{1}{x-c}$; assuming that $f(x)$ is of lower degree than 3.

1935

1. (*b*) (*i*) 0, $-\frac{5}{2}$, (*ii*) $x=\pm 3$. $y=\pm 2$

2. (*b*) (*i*) $\frac{1}{6}n(n+1)(2n+1)$;

(*ii*) $\frac{n}{18}(6n^2+15n+11)$. **3.** (*a*) $n=9$;

(*b*) $\frac{\lfloor n+1}{\lfloor n \quad \lfloor m-n+1}$. **4.** (*a*) $\frac{(n+1)(n+2)\ldots(n+r-1)}{\lfloor r-1}$;

(*b*) $\frac{1}{2}\left[\frac{1}{(x+1)^2}+\frac{1}{x+1}-\frac{x}{x^2+1}\right]$.

1936

1. (*b*) (*i*) $\frac{1}{5}$, $-\frac{1}{2}$; (*ii*) $x=12$, 9. $y=\frac{1}{2}$, $\frac{1}{3}$. **2.** (*a*) $a\frac{a^n-1}{a-1}$

$+b\frac{n(n+1)}{2}$. **3.** (*a*) $n=40$, $r=5$. (*b*) 14.

4. (*b*) $\frac{\lfloor 2n}{(\lfloor n)^2}$ or (*b*) $\frac{3}{5}\cdot\frac{1}{x-1}-\frac{1}{4}\cdot\frac{1}{x-2}-\frac{7}{20}\cdot\frac{x+2}{x^2+4}$.

1937

1. (*a*) $(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2$; (*b*) (*i*) 0, $\pm\sqrt{5}$;

(*ii*) $x=3, \frac{25}{6}$ $y=4, \frac{25}{8}$. **2.** (*a*) $^{25}\!/_{9}{}^{\circ}$; (*b*) $3>x>-1$.

3. (*i*) 462 ; (*ii*) 462. **4.** $\frac{\lfloor 2n+1}{\lfloor n+1 \quad \lfloor n}$; $\frac{\lfloor 2n}{(\lfloor n)^2}$.

1938

1. $\frac{1}{2}(p\pm\sqrt{p^2-4q})$; $np^2=(n+1)^2q$.

2. (*i*) $\left\{\frac{n(n+1)}{2}\right\}^2$, (*ii*) $\frac{2n}{n+1}$. **3.** 231 ; 371.

4. (*a*) $(1+\frac{1}{2})^{-\frac{1}{2}}=\sqrt{\frac{2}{3}}$. (*b*) 12 or $\frac{7}{5(x-3)^2}$

$+\frac{3}{25(x-3)}-\frac{3}{25(x+2)}$.

APPENDIX

Zero and Infinity.

1. *Two meanings of zero.*

The usual conception of zero is that of total absence of magnitude. It is the result of subtracting from a number, another number equal to the former one. Thus the operational definition of zero is given by the equation

$$a - a = 0,$$

where a may have any vaule whatever.

This zero is called Absolute zero, as distinguished from the zero defined otherwise in the manner explained below.

2. Consider a finite number, unity say, to be divided by a million. The result is a very small number. Raise this very small number to the millionth power. The result is a much smaller number. If this result is again raised to a millionth power, the result is still much smaller, and more nearly equal to zero. If this process is continued, the successive results approch nearer and nearer to absolute zero, though they never attain that value zero.

This is the case of a variable quantity undergoing endless decrease, and approaching the value zero.

This zero may be conceived as a number which is smaller than any assignable fraction of unity.

3. Again, let us consider the relations

$$(a+x) - a = x$$

and $$a - (a+x) = -x.$$

If we cause the x to become smaller and smaller till it finally approachs a and, therefore, if we take x to be a quantity which is less than any assignable quantity however small the above equations give

$$a - a = +0$$

$$a - a = -0.$$

The property $+0 = -0$ is possessed by no quantity other than zero.

So we see that the new meaning given to zero is that of a quantity which is smaller than any assignable quantity.

4. *Meaning of Infinity*:

Infinity may be defined as a quantity which is larger than any assignable quantity, however great.

It is not possible to think of the greatest possible number. Any number, however large we may conceive it, will be still less than infinity.

Infinity should not be taken to be like an ordinary number to which the fundamental laws of Algebra are applicable.

The statement $x = \infty$ does not mean that x gets a value ∞, but it means that x undergoes a process of endless increase or that x approaches ∞.

Note. Any quantity which is neither infinitely large, nor infinitely small is called a Finite quantity.

5. *Properties of zero and Infinity.*

	Properties of Zero	Properties of Infinity
(*i*)	Addition of zero $0+0=0$; $a+0=a$	$\infty+0=\infty$
(*ii*)	Subtraction of zero $0-0=0$; $a-0=a$.	$\infty-0=\infty$
(*iii*)	Multiplication by 0 $0\times0=0$; $a\times0=0$.	$\infty\times0$ Indeterminate (See Art 4.)
(*iv*)	Division by 0 $0\div0$ Indeterminate $0\div a=0$.	$\infty\div0=\infty$
(*v*)	Division of zero $0\div0$ Indeterminate ; $0\div a=0$.	$0\div\infty=0$.
(*vi*)	Addition of Infinity $0+\infty=\infty$; $a+\infty=\infty$	$\infty+\infty=\infty$
(*vii*)	Subtraction of Infinity $0-\infty=-\infty$; $a-\infty=-\infty$	$\infty-\infty$ Indeterminate
(*viii*)	Multiplication by Infinity $0\times\infty$ Indeterminate ; $a\div\infty=0$.	$\infty\div\infty$ Indeterminate
(*ix*)	Division by Infinity $0\div\infty=0$; $a\div\infty=0$.	$\infty\div\infty$ Indeterminate

(x) Division of Infinity

$\infty \div 0 = \infty$; $\infty \div a = \infty$; $\infty \div \infty$ Indeterminate

The signs of equality in the above statements involving ∞, should be taken by mean 'approaches' and not 'is actually equal to'.

Note. It should be clearly realised that division by Zero is not permissible under the laws of Algebra; and, therefore, we cannot remove zero from the two sides of an equality, for, then, if

$$x^2 - x^2 = x(x - x)$$

or $(x + x)(x - x) = x(x - x)$

or $x + x = x_1$, removing the factor $x - x$ which is zero.

We get $2 = 1$, which is absurd.

6. *Indeterminate Form.* Whenever a fraction assumes the form $\frac{0}{0}$, *i.e.*, when the numerator and the denominator become zero simultaneously, we get a fraction whose value is indeterminate and this value cannot be found by the ordinary rules of Algebra.

The examples of indeterminate forms are

$\frac{x^2 - 1}{x - 1}$ when $x = 1$, $\frac{x^2 + 5x + 6}{x + 2}$ when $x = -2$.

A formal discussion of these forms is beyond the scope of this book.